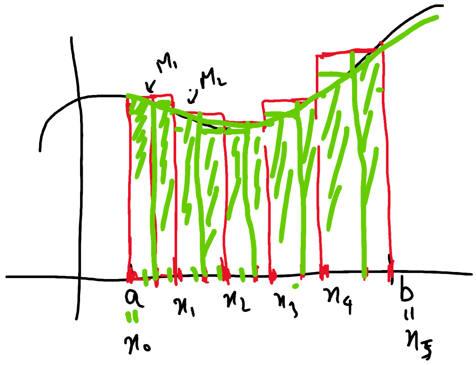
Riemann Integration

Darboux Integration

$$\underline{\int_a^b f(x)\,dx} \qquad \int f(x)\,dx$$

↑ geometric



$l \times b$

$$M_i = \max_{[x_{i-1}, x_i]} f(x) \qquad M_1 = \max_{[x_0, x_1]} f(x)$$

$$m_i = \min_{[x_{i-1}, x_i]} f(x)$$

$$U(P,f) \not\to \int_a^b f(x)\,dx$$

$$L(P,f) \not\to$$

$$L(P,f) \leq \int_a^b f\,dx \leq U(P,f)$$

$[0, \infty) \leftarrow$ ~~closed~~ open in $\mathbb{R}$

* Let $J = [a,b]$ be a closed and bounded interval and let $f : [a,b] \to \mathbb{R}$ be a bounded f^n.

Partition A partition or sub-division of $[a,b]$ is a finite set $\{x_0, x_1, x_2, \dots, x_n\}$ such that $a = x_0 < x_1 < x_2 < \dots < x_n = b$

The points x_i are called nodes of P.

Given two partitions P and Q, Q is a refinement of P if $P \subset Q$

$$m_i(f) = \inf_{x \in [x_i, x_{i+1}]} f(x) \qquad m(f) = \inf_{x \in [a,b]} f(x)$$

$$M_i(f) = \sup_{x \in [x_i, x_{i+1}]} f(x) \qquad M(f) = \sup_{x \in [a,b]} f(x)$$

$x_i \quad x_{i+1}$

$x_{i+1} - x_i$

Definition Given $f : [a,b] \to \mathbb{R}$ and a partition P of $[a,b]$

Darboux lower sum $= L(f,P) = \sum_{i=0}^{n-1} m_i(f)\,(x_{i+1} - x_i)$

Darboux Lower sum $= L(f,P) = \sum_{i=0}^{n-1} m_i(f)\,(x_{i+1}-x_i)$

Darboux Upper sum $= U(f,P) = \sum_{i=0}^{n-1} M_i(f)\,(x_{i+1}-x_i)$

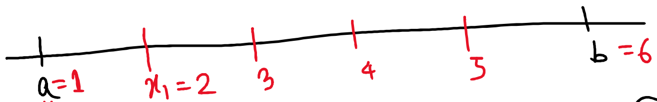
Example

1) $J=[a,b]$ $\quad P=\{a=x_0,\ b=x_n\}$

2) $J=[a,b]$ $\quad n\in\mathbb{N}$ $\quad x_i = a+\frac{i}{n}(b-a)$ $\quad \forall\ 0\le i\le n$

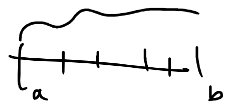
$n=5$

$a=1$ $b=6$



$x_1 = 1+\frac{1}{5}(6-1) = 1+1=2$

$x_2 = 1+\frac{2}{5}(6-1) = 1+2=3$



m $\quad m_i$

Note

1)
$$L(P,f) = \sum_{i=0}^{n-1} m_i\,(x_{i+1}-x_i)$$
$$\geq \sum_{i=0}^{n-1} m\,(x_{i+1}-x_i)$$
$$= m\sum_{i=0}^{n-1} x_{i+1}-x_i$$
$$= m\,(\cancel{x_1}-x_0+\cancel{x_2}-\cancel{x_1}+x_3-\cancel{x_2}\dots+x_n-\cancel{x_{n-1}})$$
$$= m\,(x_n-x_0) = m\,(b-a)$$

$A\subseteq B$

$m(A)\geq m(B)$

$M(A)\leq M(B)$

$$\underbrace{M(b-a)\geq U(P,f)\geq L(P,f)\geq m(b-a)}$$

2) $P=\{x_0,x_1,\dots,x_n\}$ $\quad Q=\{x_0,x_1,\dots t,\dots x_n\}$

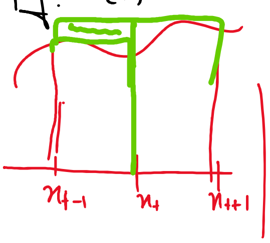
$Q = P\cup\{t\}$ $\quad P\subset Q$

Q in a refinement of P

(i) $U(P,f)\geq U(Q,f)$

(ii) $L(P,f)\leq L(Q,f)$

Pf. (i)



$$U(Q,f) = M_1(x_1-x_0)+M_2(x_2-x_1)+\dots+\underbrace{M_t(x_t-x_{t-1})+M_{t+1}(x_{t+1}-x_t)}+\dots$$
$M_n(x_n-x_{n-1})$

$$M_t(t-x_{t-1})+M_t(x_{t+1}-t)\leq M_i(t-x_{t-1})+M_i(x_{t+1}-t)$$
$$= M_i(x_{t+1}-x_t)$$

$$= M_i (x_{t+1} - x_t)$$

$$\Rightarrow \sum M_i (x_{i+1} - x_i) \leq \sum M_i (x_{i+1} - x_i)$$

$$\Rightarrow \quad U(Q, f) \leq U(P, f)$$

Def^n $U(f)$ = upper integral $= \overline{\int_a^b} f\,dx = \inf \{ U(f,P) \mid P \text{ is a partition of } [a,b] \}$

$L(f)$ = Lower integral $= \underline{\int_a^b} f\,dx = \sup \{ L(f,P) \mid P \text{ is a partition of } [a,b] \}$

Def^n f is Darboux integrable (integrable) if $U(f) = L(f)$.

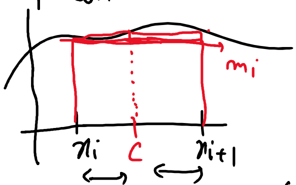
17/02/2024

Thm Let $f : [a,b] \to \mathbb{R}$ be a bounded f^n. Suppose P and Q are partitions of $[a,b]$. Then we have the following:

(i) If Q is a refinement of P, then $L(f,P) \leq L(f,Q)$ and $U(f,P) \geq U(f,Q)$

(ii) $L(f,P) \leq U(f,Q)$ for any two partitions P and Q.

(iii) $L(f) \leq U(f)$



Proof. (i) ATQ,

$$L(f,P) = \sum_{i=0}^{n-1} m_i (x_{i+1} - x_i) = m_0 (x_1 - x_0) + \cdots + m_i (x_{i+1} - x_i) + \cdots + m_{n-1} (x_n - x_{n-1})$$

$$L(f,Q) = m_0 (x_1 - x_0) + \cdots + m_{c_1} (c - x_i) + m_{c_2} (x_{i+1} - c) + \cdots + m_{n-1} (x_n - x_{n-1})$$

where $m_{c_1} = \inf\limits_{x \in [x_i, c]} f(x)$ and $m_{c_2} = \inf\limits_{x \in [c, x_{i+1}]} f(x)$

We have $m_{c_1} \geq m_i$ and $m_{c_2} \geq m_i$

$$\therefore L(f,Q) \geq m_0 (x_1 - x_0) + \cdots + m_i (c - x_i) + m_i (x_{i+1} - c) + \cdots + m_{n-1} (x_n - x_{n-1})$$

$$\Rightarrow L(f,Q) \geq m_0 (x_1 - x_0) + \cdots + m_i (x_{i+1} - x_i) + \cdots + m_{n-1} (x_n - x_{n-1}) = L(f,P)$$

Hint: $M_{c_1} = \sup\limits_{x \in [x_i, c]} f(x) \qquad M_{c_2} = \sup\limits_{x \in [c, x_{i+1}]} f(x)$

$$M_{c_1} \leq M_i \qquad M_{c_2} \leq M_i$$

$\cancel{L(f,P) \leq L(f,Q)}$.

$P \subseteq Q$