11/03/24

Defn Let $f : (a,b) \to \mathbb{R}$. Then f is absolutely integrable on $[a,b]$ if $|f|$ is integrable on $[a,b]$.

f is said to be conditionally integrable on $[a,b]$ if f is integrable on $[a,b]$ but not absolutely integrable.

$|f|$ – integrable $\qquad f \to$ not integrable ?

Example

$$|f|(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}$$

$$U(P,f) = \sum_{i=0}^{n-1} 1 \cdot \Delta x = x_n - x_0 = b-a$$

$$\neq$$

$$L(P,f) = \sum -1 \cdot \Delta x = a-b$$

$[a,b]$ $\qquad a \to -\infty,\ b \to \infty$ $\qquad f$ bounded

Improper Riemann integrals

We extend the concept of Riemann integrals to functions

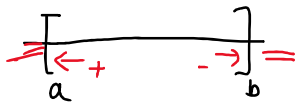
(i) f^n defined on unbounded intervals

(ii) unbounded f^n

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$$

Proposition Let $f : [a,b] \to \mathbb{R}$ be integrable. Then

$$\int_a^b f(t)\,dt = \lim_{c \to a+} \left(\lim_{d \to b-} \int_c^d f(t)\,dt \right)$$



$$\int_a^b f(t)\,dt = \lim_{c\to a+}\left(\lim_{d\to b-}\int_c^d \cdots\right)$$

Pf. Let $g(x) := \int_a^x f(t)\,dt$. Then g is cont^n on $[a,b]$. Therefore

$$\int_a^b f(t)\,dt = g(b) - g(a)$$
$$= \lim_{c\to a+}\left(\lim_{d\to b^-}[g(d) - g(c)]\right)$$

Def^n. Let $(a,b) \neq \emptyset$, possibly unbounded and

$(a,a) \neq \emptyset$

$\therefore [a,a] = \{a\}$

$$f : (a,b) \to \mathbb{R}$$

(i) We say that f is locally integrable on (a,b) if f is integrable on each closed sub-interval $[c,d]$ of (a,b).

(ii) We say that the improper Riemann integral of f exists or is convergent on (a,b) if

$$\lim_{c\to a+}\left(\lim_{d\to b^-}\int_c^d f(t)\,dt\right)$$

exists. The limit is denoted by $\int_a^b f(t)\,dt$.

Proposition The order in which limits are taken in the last def^n does not matter.

Proof Let $t_0 \in (a,b)$ be fixed. Then

$$\lim_{c\to a+}\left(\lim_{d\to b-}\int_c^d f(t)\,dt\right) = \lim_{c\to a+}\left(\int_c^{t_0} f(t)\,dt + \lim_{d\to b-}\int_{t_0}^d f(t)\,dt\right)$$
$$= \lim_{c\to a+}\int_c^{t_0} f(t)\,dt + \lim_{d\to b-}\int_{t_0}^d f(t)\,dt$$
$$= \lim_{d\to b-}\left(\lim_{c\to a+}\int_c^{t_0} f(t)\,dt + \int_{t_0}^d f(t)\,dt\right).$$
$$= \lim_{d\to b-}\left(\lim_{c\to a+}\int_c^d f(t)\,dt\right)$$

$$= \lim_{d \to b-} \left(\lim_{c \to a+} \int_c f(t)\,dt \right)$$

Notation
$$\lim_{\substack{c \to a+ \\ d \to b-}} \int_c^d f(t)\,dt = \int_a^b f(t)\,dt$$

Examples: Check if the following improper integrals exists / converge:

(i) $\int_0^1 x^{-1/2}\,dx$ (ii) $\int_0^\infty e^{-x}\,dx$

Sol^n (i) $(0,1]$

$f(x) = x^{-1/2}$

$f(0) = 0^{-1/2} = \infty$

$f(1) = 1$

$(0,1]$

$$\lim_{c \to 0} \int_c^1 x^{-1/2}\,dx = \lim_{c \to 0} \left[\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right]_c^1$$

$$= \lim_{c \to 0} 2\left[x^{1/2}\right]_c^1$$

$$= \lim_{c \to 0} 2\left[1 - c^{1/2}\right] = 2 \cdot 1 = 2$$

$\therefore \int_0^1 x^{-1/2}$ exists and is equal to 2.

(ii) $\int_0^\infty e^{-x}\,dx$ $[0,\infty)$ $\cancel{[0,\infty)}$

$$\lim_{d \to \infty} \int_0^d e^{-x}\,dx = \lim_{d \to \infty} \left[-e^{-x}\right]_0^d$$

$$= \lim_{d \to \infty} -\left[e^{-d} - 1\right]$$

$$= 0 + 1 = 1$$

(iii) $\int_0^1 \frac{dx}{x(x-1)}$ $(0,1)$

$f(x) = \frac{1}{x(x-1)}$

$f(0) = \frac{1}{0(-1)} = \frac{1}{0}$

$f(1) = \infty$

$$\frac{1}{x(x-1)} = -\frac{1}{x} + \frac{1}{x-1}$$

$$\frac{1}{x-1} - \frac{1}{x} = \frac{x-(x-1)}{\;}$$

$\int^1 dx$ $\quad \int^1 dx$ $\quad \int^1 dx$

$$\int_0^1 \frac{dx}{x(x-1)} = \int_0^1 \frac{dx}{x-1} - \int_0^1 \frac{dx}{x} \qquad \frac{1}{x-1} - \frac{1}{x} = \frac{x-(x-1)}{\;} = \frac{1}{\;}$$

$$= \lim_{d\to 1} \int_0^d \frac{dx}{x-1} - \lim_{c\to 0} \int_c^1 \frac{dx}{x}$$

$$= \lim_{d\to 1} \left[\ln|x-1|\right]_0^d -$$

$$= \lim_{d\to 1} \left[\ln|d-1| - \ln|1|\right]$$

$$= \lim_{d\to 1} \ln|d-1| -$$

Thm ⊛ If f is absolutely integrable on (a,b) then the improper integral of f on (a,b) exists and we have

$$\left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)|\,dt.$$

Thm (Integral test) Assume that $f:[1,\infty) \to [0,\infty)$ is continuous and decreasing. Let $a_n = f(n)$ and $b_n = \int_1^n f(t)\,dt$. Then

(i) $\sum a_n$ converges if the improper integral $\int_1^\infty f(t)\,dt$ exists.

(ii) $\sum a_n$ diverges " " " " $\int_1^\infty f(t)\,dt$ doesnot exist.

Proof (i) $a_n = f(n)$ $\qquad b_n = \int_1^n f(t)\,dt$

$$\sum a_n = \sum f(n) \qquad \sum b_n = \sum_{n=1}^{\infty} \int_1^n f(t)\,dt$$

$$= \int_1^2 f(t)\,dt + \int_2^3 + \dots \int^\infty$$

$$= \int_1^\infty f(t)\,dt$$

Series comparison test

$a_n \leq b_n \quad n \geq N_0$

① $\sum a_n$ converge if $\sum b_n$ conver

②

$$a_n \leq b_n \quad \forall\, n \in \mathbb{N}$$

$\sum a_n$ converges if $\sum b_n$ converges

$a_n \le b_n$

$\therefore$ $\sum a_n$ converges if $\sum b_n$ converges

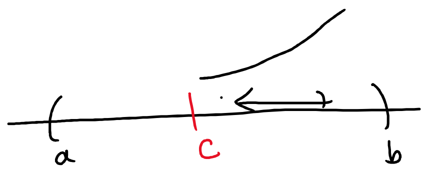
" " if $\int_1^\infty f(t)\,dt$ "

Thm (Comparison test)

Proof Fix $c \in (a,b)$. Let

$F(d) = \int_c^d f(t)\,dt$ and $\quad d \in [c,b)$

$G(d) = \int_c^d g(t)\,dt$



$\because$ $f \geq 0$, the fn F is increasing on $[c,b)$

Hence $F(b-)$ exists. Thus the improper integral of f exists on (c,b) and we get

$$\int_c^b f(t)\,dt = F(b-) \le G(b-) = \int_c^b g(t)\,dt \quad -(1)$$

Similarly $$\int_a^c f(t)\,dt \le \int_a^c g(t)\,dt \quad -(2)$$

(1) + (2) $\Rightarrow$

$$\int_a^b f(t)\,dt \le \int_a^b g(t)\,dt$$

Proof of (*) $\because$ $f(x) \le |f(x)|$ $\forall$ x $\qquad \int |f(x)|\,dx$

$\therefore$ f is locally integrable on (a,b)

$|f(x)| \;\Big\langle\; \begin{array}{l} 2f(x) \\ + f(x) + f(x) \\ - f(x) + f(x) \\ 0 \end{array}$

We have $0 \le f(x) + |f(x)| \le 2|f(x)|$ $\qquad |f(x)| + f(x)$

$\because$ $\int^b |f(x)|\,dx$ exists, therefore

We have [illegible]

$\because \int_a^b |f(x)|\,dx$ exists, therefore

$$\int_a^b \left(f(x) + |f(x)| \right) dx \text{ exists} \quad \text{(By comparison)}$$

$\because f = \underline{f + |f|} - \underline{|f|}$, therefore improper integral of f exists and we have

$$\left| \int_a^b f(t)\,dt \right| \leq \int_a^b |f(t)|\,dt.$$