Example: Investigate the behaviour of the solution of the differential equation (which is the diff. eqn. of limited population growth with harvesting)

$$\frac{du}{dt} = ru\left(1 - \frac{u}{K}\right) - h$$

when $r = 1$; $K = 10$, $h = \frac{9}{10}$ and $u(0) = u_0$.

The symbols have their usual meanings.

Also solve the differential eqn.

Soln. Given diff. eqn. is

$$\frac{du}{dt} = ru\left(1 - \frac{u}{K}\right) - h \longrightarrow (1)$$

Also given,

$$r = 1$$
$$K = 10$$
$$h = \frac{9}{10}$$
$$\& \quad u(0) = u_0.$$

Now, (1) $\Rightarrow$

$$\frac{du}{dt} = ru\left(\frac{K-u}{K}\right) - h$$

$$= \frac{r}{K}\left(Ku - u^2 - \frac{Kh}{r}\right)$$

$$\Rightarrow \frac{du}{dt} = -\frac{r}{K}\left(u^2 - Ku + \frac{Kh}{r}\right)$$

$$= -\frac{1}{10}\left[u^2 - 10u + \frac{10 \times \frac{9}{10}}{1}\right]$$

$$= -\frac{1}{10}\left[u^2 - 10u + 9\right]$$

$$= -\frac{1}{10}\left[(u-1)(u-9)\right] \longrightarrow (2)$$

Case 1. If $u < 1$, then $u'(t) < 0$

$\Rightarrow$ population declines.

Case 2. If $1 < u < 9$, then $u'(t) > 0$

$\Rightarrow$ Population increases.

Case 3. If $u > 9$, then $u'(t) < 0$

$\Rightarrow$ Population declines.

Case 4. If $u = 1$ or $u = 9$, then $u'(t) = 0$.

$\Rightarrow u(t) = \text{constant}$

$\Rightarrow$ population does not change.

To solve the given diff. eqn:

We have from (2),

$$\frac{du}{dt} = -\frac{1}{10}(u-1)(u-9).$$

$$\Rightarrow \frac{du}{(u-1)(u-9)} = -\frac{1}{10}dt$$

$$\Rightarrow \int \frac{1}{8}\left[\frac{1}{u-9} - \frac{1}{u-1}\right] du = -\frac{1}{10}\int dt +$$

$$\Rightarrow \frac{1}{8}\left[\log_e(u-9) - \log_e(u-1)\right] = -\frac{1}{10}t +$$

$$\Rightarrow \frac{1}{8}\log\left|\frac{u-9}{u-1}\right| = -\frac{1}{10}t + c$$

$$\Rightarrow \log\left|\frac{u-9}{u-1}\right| = -\frac{4}{5}t + 8c \longrightarrow (3)$$

Now, using initial condn $u(0) = u_0$ in (3), we get

$$\log\left|\frac{u_0-9}{u_0-1}\right| = 8c$$

$$\therefore (3) \Rightarrow \log\left|\frac{u-9}{u-1}\right| = -\frac{4}{5}t + \log\left|\frac{u_0-9}{u_0-1}\right|$$

$$\Rightarrow \log\left|\frac{u-9}{u-1} \times \frac{u_0-1}{u_0-9}\right| = -\frac{4}{5}t$$

$$\Rightarrow \frac{u-9}{u-1} \times \frac{u_0-1}{u_0-9} = e^{-\frac{4}{5}t}$$

$$\Rightarrow \frac{u-9}{u-1} = \frac{u_0-9}{u_0-1}e^{-\frac{4}{5}t}$$

$$\Rightarrow u-9 = (u-1)\frac{u_0-9}{u_0-1}e^{-\frac{4}{5}t}$$

$$= (u-1)Ae^{-4/5t}, \quad A = \frac{u_0-9}{u_0-1}$$

$$\Rightarrow u-9 = uAe^{-\frac{4}{5}t} - Ae^{-\frac{4}{5}t}$$

$$\Rightarrow u - uAe^{-\frac{4}{5}t} = 9 - Ae^{-\frac{4}{5}t}$$

$$\Rightarrow u\left[1 - Ae^{-\frac{4}{5}t}\right] = 9 - Ae^{-\frac{4}{5}t}$$

$$\Rightarrow u(t) = \frac{9 - Ae^{-\frac{4}{5}t}}{1 - Ae^{-\frac{4}{5}t}} \longrightarrow (4)$$

which is the required sol^n.

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