## Density dependent population growth model :→

[ **Background:** As a population grows, individuals eventually will compete for the limited resources available. In principle, this competition means that a given environment can support only a limited number of individuals. This number is called carrying capacity for the population and usually denoted by the symbol K in biological literature. Technically, we define it as the population size (or density) for which the per-capita birth rate is equal to the per-capita death rate, excluding external factors such as harvesting or interection with another population. We need to extend the model to include an additional death rate due to the resource limitations.

In reality, population cannot continue growing exponentially over time due to limited resources or competistion for these with other species. If we observe the population over long period, they reach a limit, or stabilize. ]

# Limited growth population model

(OR)

## Density Dependent population growth model :-

In density dependent growth model, we account for increase in death rate due to crowding and competition for limited resources and stabilizing effect of population after a long period. After a long period, population stabilizes rather than growing exponentially.

By balance law, word eqn. can be written as

$$\left\{\begin{matrix}\text{rate of change}\\ \text{in population}\end{matrix}\right\} = \left\{\begin{matrix}\text{rate of}\\ \text{births}\end{matrix}\right\} - \left\{\begin{matrix}\text{normal}\\ \text{rate of}\\ \text{deaths}\end{matrix}\right\} - \left\{\begin{matrix}\text{rate of}\\ \text{deaths}\\ \text{by density}\end{matrix}\right\}$$

We make the following assumptions:

(i) Population is sufficiently large so that we can ignore random differences bet$^n$. individuals

(ii) The births & deaths are cont$^s$. in time.

(iii) Per-capita birth & death rates are cont$^s$. in time.

(iv) We ignore immigration and emigration.

## Formulating the differential Equation :-

Let $u(t)$ be the population in time $t$ and let us consider a population whose initial value is $u_0$, with constant per-capita birth rate $\beta$ and const. per-capita death rate $\alpha$. Also let $\gamma u$ be the per-capita death rate due to density dependent death, as it is assumed to be proportional to population size.

Now we have,

Birth rate at any time t is $= \beta u(t)$,

Death rate at any time t is $= \alpha u(t)$.

Also, Deaths due to crowding = (per capita death rate due to density) x (current population size.

$\therefore$ Rate of deaths due to density $= (\gamma u)u = \gamma u^2$

Hence, word eqn. transforms to differential eqn.

$$\frac{du}{dt} = \beta u - \alpha u - \gamma u^2 = (\beta - \alpha) u - \gamma u^2$$

$$\Rightarrow \frac{du}{dt} = r u - \gamma u^2, \text{ where } \beta - \alpha = r.$$

$$\Rightarrow \frac{du}{dt} = r u \left(1 - \frac{\gamma u}{r}\right)$$

$$\Rightarrow \frac{du}{dt} = r u \left(1 - \frac{u}{K}\right)$$

where, $K = \frac{r}{\gamma}$ is carrying capacity.

At initial condn, $u(0) = u_0$.

Hence, IVP for density-dependent growth model is

$$\boxed{\frac{du}{dt} = r u \left(1 - \frac{u}{K}\right) \;;\; u(0) = u_0}$$

This model is also known as logistic model or limited growth population model.

<u>Ex.</u> Solve the logistic differential Equation

$$\frac{du}{dt} = ru\left(1-\frac{u}{K}\right) \quad \text{with } u(0)=u_0.$$

<u>Sol<sup></sup></u>: Given diff eqn is

$$\frac{du}{dt} = ru\left(1-\frac{u}{K}\right) \longrightarrow (1)$$

$$\Rightarrow \frac{K}{u(K-u)}\,du = r\,dt$$

$$\Rightarrow \int\left[\frac{1}{u}+\frac{1}{K-u}\right]du = \int r\,dt + \log_e c, \quad c \to \text{const.}$$

$$\Rightarrow \log_e u - \log_e(K-u) = rt + \log_e c$$

$$\Rightarrow \frac{u}{c(K-u)} = e^{rt}$$

$$\Rightarrow u(t) = c(K-u)e^{rt} \longrightarrow (2)$$

Using the given initial condn $u(0)=u_0$, in (2), we get

$$u_0 = c(K-u_0)e^0$$

$$\Rightarrow c = \frac{u_0}{K-u_0}$$

$$\therefore (2) \Rightarrow u(t) = \frac{u_0}{K-u_0}(K-u)e^{rt}$$

$$\Rightarrow u + \frac{u_0}{K-u_0}ue^{rt} = \frac{Ku_0}{K-u_0}e^{rt}$$

$$\Rightarrow u\left[K-u_0+u_0e^{rt}\right] = u_0Ke^{rt}$$

$$\Rightarrow u\left[\frac{K-u_0}{u_0}e^{-rt}+1\right] = K$$

$$\Rightarrow u = \frac{K}{1+c_1e^{-rt}}, \quad \text{where } c_1 = \frac{K-u_0}{u_0}$$

which is the soln. of the given logistic differential equation (i.e. of limited growth of population model)

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## Equilibrium solution, equilibrium points and stability of a solⁿ.

If the solⁿ of a differential equation tends to a constant value i.e. to a steady state, when the time become very large, then the solⁿ is called equilibrium solution.

For a differential equation

$$\frac{du(t)}{dt} = f(u), \longrightarrow (1)$$

the equilibrium solutions are the solutions $u = u_e$ such that $f(u) = 0$ i.e. $\frac{du(t)}{d(t)} = u'(t) = 0$.

The values of $u$ for which $f(u) = 0$, are called the equilibrium points. (i.e. solving $f(u) = 0$, we get the equilibrium points.)

Stability :-

A stable solution is a solution close to the equilibrium solution which tends towards the equilibrium solution (as the time increases without bound). On the other hand if the solⁿ do not get closer to the equilibrium solⁿ (or, point), then it is called an unstable solⁿ.

Using Tailor series expansion in the differential eqn (1) near a equilibrium point, it can be shown that –

(i) equilibrium solⁿ is stable if $f'(u_0) < 0$

and (ii) " " " unstable if $f'(u_0) > 0$.