Model for Exponential growth of Population :-

(Exponential population growth model)

We consider exponential growth model as a compartmental model, with the compartment being the 'world', 'town', 'organization', 'ocean' etc. Here the compartmental diagram is

$$\xrightarrow{\text{births}} \boxed{\text{world}} \xrightarrow{\text{deaths}}$$

The word eqn. which describes the changing of a population is given by

$$\left\{\begin{array}{c}\text{rate of} \\ \text{change of} \\ \text{Population size}\end{array}\right\} = \left\{\begin{array}{c}\text{rate} \\ \text{of} \\ \text{births}\end{array}\right\} - \left\{\begin{array}{c}\text{rate} \\ \text{of} \\ \text{deaths}\end{array}\right\}$$

We make the following assumptions :-

(i) We assume that the population are sufficiently large so that we can ignore random differences between individuals.

(ii) We assume that per-capita birth death rates are constant in time

(ii) We assume that the birth and death are continuous in time.

(iii) In the model developed we ignore immigration and emigration.

Formulating the differential Equation :-

Let us consider a population whose initial value is u_0, with constant per-capita birth rate given as β, and constant per-capita death rate given as α.

Let $u(t)$ be the population at time t.

Since β is constant,

$\therefore$ The birth rate at any time $= \beta u(t)$.

Also since α is constant,

$\therefore$ The death rate at any time $= \alpha\, u(t)$.

(*) Again, we assume that the rate of change in population at any time is proportional to the size of the population at that time.

Hence by balance law, the word equation transforms to the differential equation

$$\frac{du(t)}{dt} = \beta u(t) - \alpha u(t)$$

i.e. $$\frac{du}{dt} = \beta u - \alpha u$$

Initially, $u(0) = u_0$.

Hence IVP for exponential growth model is given by

$$\boxed{\frac{du}{dt} = \beta u - \alpha u, \quad u(0) = u_0} \longrightarrow (1)$$

Now, (1) $\Rightarrow$

$$\frac{du}{dt} = (\beta - \alpha) u = r u \longrightarrow (2)$$

where $r = \beta - \alpha$ and r is called growth rate or the reproduction rate for the population. When $r > 0$, this is a model describing exponential growth, and when $r < 0$, then it describes exponential decay model.

Now, from (2),

$$\frac{du}{u} = r\,dt$$

Integrating, $\log u = rt + \log c$, c is const

$$\Rightarrow u(t) = c\,e^{rt} \longrightarrow (3).$$

Using initial condition, $u(0) = u_0$, we get $c = u_0$

Hence from (3), putting $c = u_0$, we get

$$\boxed{u(t) = u_0 e^{rt}} \longrightarrow (4)$$

which is the soln. of (1).

Ex Find an expression for the time for a population to double in size.

Sol^n. We have model for exponential growth of population

$$\frac{du(t)}{dt} = ru, \quad u(0) = u_0 \longrightarrow (1)$$

where $u(t)$ is the population in time t and r is the reproduction rate for the population.

Now, (1) $\Rightarrow$

$$\frac{du(t)}{u} = r\,dt$$

Integrating, we get

$$\log_e u = rt + \log c, \quad c \to \text{is constant}$$

$$\Rightarrow u(t) = c\,e^{rt} \longrightarrow (2)$$

which is the G. sol^n. of (1).

Using the initial condition $u(0) = u_0$, we get

$$u_0 = c\,e^0 = c$$

$\therefore$ Putting $c = u_0$ in (2), we get

$$u = u_0 e^{rt} \longrightarrow (3)$$

which is the sol^n of (1) with given initial cond^n.

Now, ATQ, we replace u by $2u_0$ when $t = T$ in (3).

$$\therefore \quad 2u_0 = u_0 e^{rT}$$

$$\Rightarrow 2 = e^{rT}$$

$$\Rightarrow rT = \log_e 2$$

$$\Rightarrow \boxed{T = \frac{1}{r}\log_e 2} \longrightarrow (4)$$

which is the time required for a population to be doubled in size.