Jaccobi Iteration Method :
(or, Gauss – Jaccobi Iteration Method)
(or, method of simultaneous displacement)

We consider the eqns.

$$\left.\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ a_2 x + b_2 y + c_2 z &= d_2 \\ a_3 x + b_3 y + c_3 z &= d_3 \end{aligned}\right\} \rightarrow (1)$$

Let a_1, b_2, c_3 be large as compared to other co-efficients in the respective eqns. (absolute value), then the system can be written as :

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \\ y &= \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \\ z &= \frac{1}{c_3}(d_3 - a_3 x - c_3 z) \end{aligned}\right\} \rightarrow (2)$$

If we start with the initial approximations x_0, y_0, z_0 for the values of x, y, z respectively, then substituting these on the right sides of (2), the first approximations $x^{(1)}, y^{(1)}, z^{(1)}$ are given by

$$\left.\begin{aligned} x^{(1)} &= \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0) \\ y^{(1)} &= \frac{1}{b_2}(d_2 - a_2 x_0 - c_2 z_0) \\ z^{(1)} &= \frac{1}{c_3}(d_3 - a_3 x_0 - b_3 y_0) \end{aligned}\right\} \rightarrow (3)$$

Again, substituting the values $x^{(1)}, y^{(1)}, z^{(1)}$ on the right hand sides of (2), we get the second approximations as

$$\left.\begin{aligned} x^{(2)} &= \frac{1}{a_1}\left(d_1 - b_1 y^{(1)} - c_1 z^{(1)}\right) \\ y^{(2)} &= \frac{1}{b_2}\left(d_2 - a_2 x^{(1)} - c_2 z^{(1)}\right) \\ z^{(2)} &= \frac{1}{c_3}\left(d_3 - a_3 x^{(1)} - b_3 y^{(1)}\right) \end{aligned}\right\} \quad (4)$$

We continue this process till the difference between two consecutive approximations is as small as we please.

Note: ①: This method can be applied to systems having more than three unknown.

Note (2): This method converges for any choice of the initial approximations if in each equation of the system, the absolute value of the largest co-efficient is greater than the sum of the absolute values of all the remaining co-efficients. i.e. in the above case, if

$$\left\{\begin{aligned} &|a_1| > |b_1| + |c_1|, \\ &|b_2| > |a_2| + |c_2| \\ \text{and } &|c_3| > |a_3| + |b_3|. \end{aligned}\right.$$

=x=

Ex. Solve, by Jaccobi Iteration method, the eqns.

$$20x+y-2z=17$$

$$3x+20y-z=-18$$

$$2x-3y+20z=25$$

Solⁿ: [[Here we see that,

In first eqn. $\underset{\text{co-effts. of } x}{|20|} > \underset{\text{co-effts. of } y}{|1|} + \underset{\text{co-effts. of } z}{|(-2)|}$,

in 2nd eqn. $\underset{\text{co-effts. of } y}{|20|} > \underset{\text{co-effts. of } x}{|3|} + \underset{\text{co-effts. of } z}{|-1|}$

in 3rd eqn. $\underset{\text{co-effts. of } z}{|20|} > \underset{\text{co-effts. of } x}{|2|} + \underset{\text{co-effts. of } y}{|-3|}$.

Hence Jaccobi method will be converge, i.e. this method can be applied here]]

We have from given system

$$\left.\begin{aligned} x &= \frac{1}{20}[17-y+2z] \\ y &= \frac{1}{20}[-18-3x+z] \\ z &= \frac{1}{20}[25-2x+3y] \end{aligned}\right\} \rightarrow (1)$$

Let us start by an approximation

$x=x_0,\ y=y_0,\ z=z_0$ & $x_0=y_0=z_0=0$.

Substituting these on the right side of (1), we get the first approximations as

$$\left.\begin{aligned} x^{(1)} &= \frac{17}{20} = 0.85 \\ y^{(1)} &= \frac{-18}{20} = -0.9 \\ z^{(1)} &= \frac{25}{20} = 1.25 \end{aligned}\right\} \longrightarrow (2)$$

Again putting these values of $x^{(1)}, y^{(1)}, z^{(1)}$ on the right hand side of (1), we obtain the second approximations as

$$\left.\begin{aligned}
x^{(2)} &= \frac{1}{20}\left(17 - y^{(1)} + 2z^{(1)}\right) \\
&= \frac{1}{20}\left(17 + 0.9 + 2(1.25)\right) = 1.02 \\
y^{(2)} &= \frac{1}{20}\left\{-18 - 3x^{(1)} + z^{(1)}\right\} \\
&= \frac{1}{20}\left\{-18 - 3(0.85) + 1.25\right\} = -0.965 \\
z^{(2)} &= \frac{1}{20}\left\{25 - 2(0.85) + 3\cdot(-0.9)\right\} = 1.515
\end{aligned}\right\} \rightarrow (3)$$

Further, substituting these values on the right side of (1), we get the 3rd approximations as:

$$\left.\begin{aligned}
x^{(3)} &= \frac{1}{20}\left\{17 - y^{(2)} - 2z^{(2)}\right\} = \frac{1}{20}\left\{17 + 0.965 + 2(1.515)\right\} = 1.0134 \\
y^{(3)} &= \frac{1}{20}\left\{-18 - 3x + z\right\} = \frac{1}{20}\left\{-18 - 3x^{(2)} + z^{(2)}\right\} \\
&= \frac{1}{20}\left\{-18 - 3(1.02) + 1.515\right\} = -0.9954 \\
z^{(3)} &= \frac{1}{20}\left\{25 - 2x^{(2)} + 3y^{(2)}\right\} = \frac{1}{20}\left\{25 - 2(1.02) + 3(-0.965)\right\} = 1.0032
\end{aligned}\right\} (4)$$

Again, putting these values in (1), we get the 4th approx. as

$$\left.\begin{aligned}
x^{(4)} &= \frac{1}{20}\left\{17 - y^{(3)} - 2z^{(3)}\right\} = \frac{1}{20}\left\{17 + 0.9954 - 2(1.0032)\right\} = 1.009 \\
y^{(4)} &= \frac{1}{20}\left\{-18 - 3x^{(3)} + z^{(3)}\right\} = \frac{1}{20}\left\{-18 - 3(1.0134) + 1.0032\right\} = -1.0018 \\
z^{(4)} &= \frac{1}{20}\left\{25 - 2x^{(3)} + 3y^{(3)}\right\} = \frac{1}{20}\left\{25 - 2(1.0134) - 3(0.9954)\right\} = 0.9993
\end{aligned}\right\} (5)$$

Putting these values of $x^{(4)}, y^{(4)}, z^{(4)}$ in (1), we obtain the 5^{th} approximation as :

$$\left.\begin{aligned} x^{(5)} &= \frac{1}{20}\{17 - y^{(4)} + 2z^{(4)}\} = \frac{1}{20}\{17 - 1.0018 + 2(1.0032)\} = 1.0000 \\ y^{(5)} &= \frac{1}{20}\{-18 - 3x^{(4)} + z^{(4)}\} = \frac{1}{20}\{-18 - 3(1.0009) + 0.9993\} = -1.0002 \\ z^{(5)} &= \frac{1}{20}\{25 - 2x^{(4)} + 3y^{(4)}\} = \frac{1}{20}\{25 - 2(1.0009) + 3(-1.0018)\} = 0.9996 \end{aligned}\right\} (6)$$

Similarly, we have 6^{th} approximations as :

$$\left.\begin{aligned} x^{(6)} &= \frac{1}{20}\{17 + 1.0002 + 2(0.9996)\} = 1.0000 \\ y^{(6)} &= \frac{1}{20}\{-18 - 3(1) + 0.9996\} = -1.0000 \\ z^{(6)} &= \frac{1}{20}\{25 - 2(1) + 3(-1.0002)\} = 1.0000 \end{aligned}\right\} \rightarrow (7)$$

So the values in the 5^{th} and 6^{th} iterations are practically the same, hence we stop here.

Thus the solⁿ is $x = 1,\ y = -1,\ z = 1$

Q. Apply Jaccobi iteration method to solve

$$10x + y + z = 12$$
$$2x + 10y + z = 13,$$
$$2x + 2y + 10z = 14.$$

Ans: $x = y = z = 1$.