Ex. Solve the system of equations

$$x_1 + 2x_2 + 3x_3 = 14$$
$$2x_1 + 5x_2 + 2x_3 = 18$$
$$3x_1 + x_2 + 5x_3 = 20$$

by LU Decomposition method (i.e., by factorization method)

Solⁿ. The given system of eqns. can be written in the matrix form as

$$AX = B$$

or,
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 14 \\ 18 \\ 20 \end{pmatrix}$$

Now let $A = LU$, where L is the lower triangular matrix and U is the upper triangular matrix. Thus we have

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

Equating the corresponding elements, we get-

$u_{11} = 1, \quad u_{12} = 2, \quad u_{13} = 3,$

Imp.
(i) 1st row
(ii) 1st col^n.
(iii) 2nd row
iv) 2nd col^n
v) 3rd row

$l_{21}u_{11} = 2 \Rightarrow l_{21} = \frac{2}{u_{11}} = \frac{2}{1} = 2,$

$l_{31}u_{11} = 3 \Rightarrow l_{31} = \frac{3}{u_{11}} = \frac{3}{1} = 3,$

$l_{21}u_{12} + u_{22} = 5 \Rightarrow 2.2 + u_{22} = 5 \Rightarrow u_{22} = 1,$

$l_{21}u_{13} + u_{23} = 2 \Rightarrow 2.3 + u_{23} = 2 \Rightarrow u_{23} = -4,$

$l_{31}u_{12} + l_{32}u_{22} = 1 \Rightarrow 3.2 + l_{32}.1 = 1 \Rightarrow l_{32} = -5,$

and $l_{31}u_{13} + l_{32}u_{23} + u_{33} = 5 \Rightarrow 3.3 + (-5).(-4) + u_{33} = 5$

$\Rightarrow u_{33} = -24.$

Hence the given eq^n $AX = B$ is equivalent to

$LUX = B \quad , (\because A = LU)$

i.e. $$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -24 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 18 \\ 20 \end{bmatrix}$$

Now, let $UX = Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\because LUX = B$
$\Rightarrow LY = B$

$\Rightarrow LY = B$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 18 \\ 20 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ 2y_1 + y_2 \\ 3y_1 - 5y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 18 \\ 20 \end{bmatrix}$$

$$\Rightarrow \begin{cases} y_1 = 14 & \longrightarrow (i) \\ 2y_1 + y_2 = 18 & \longrightarrow (ii) \\ 3y_1 - 5y_2 + y_3 = 20 & \longrightarrow (iii) \end{cases}$$

$(ii) \Rightarrow y_2 = 18 - 2y_1 = 18 - 2.14 = -10$

$(iii) \Rightarrow y_3 = 20 - 3y_1 + 5y_2 = 20 - 3(14) + 5(-10)$

$= 20 - 42 - 50$

$= -72$

Now, $UX = Y$ gives

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -24 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -10 \\ -72 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 14 & \rightarrow (iv) \\ x_2 - 4x_3 = -10 & \longrightarrow (v) \\ -24x_3 = -72 & \rightarrow (vi) \end{cases}$$

$(vi) \Rightarrow x_3 = 3$

$(v) \Rightarrow x_2 = -10 + 4x_3 = -10 + 4(3) = 2$

$(iv) \Rightarrow x_1 = 14 - 2x_2 - 3x_3 = 14 - 2(2) - 3(3) = 1$

Hence the solution of the given eqns is

$$\left. \begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 3 \end{aligned} \right\} \longleftarrow \text{Ans.}$$

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H.W Ex. Solve the following system of eqns by LU Decomposition method (i.e. by factorization method).

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(1)

$$2x - 3y + 10z = 3$$
$$-x + 4y + 2z = 20$$
$$5x + 2y + z = -12$$

Ans: $\begin{cases} x = -4 \\ y = 3 \\ z = 2 \end{cases}$

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(2)

$$5x - 2y + z = 4$$
$$7x + y - 5z = 8$$
$$3x + 7y + 4z = 10$$

Ans: $\begin{cases} x = \frac{122}{109} \\ y = \frac{284}{372} \\ z = \frac{46}{327} \end{cases}$

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(3)

$$2x + 3y + z = 9$$
$$x + 2y + 3z = 6$$
$$3x + y + 2z = 8$$

Ans: $\begin{cases} x = \frac{35}{18} \\ y = \frac{29}{18} \\ z = \frac{5}{18} \end{cases}$

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