Solution of Simultaneous Linear Algebric Equations:

LU Decomposition method
(OR, Doolittle method)
(or, Method of Factorization)

Let the system of eqn. in matrix form be

$$AX = B$$

where A is n^{th} order square matrix, X is $n \times 1$ matrix and B is $n \times 1$ matrix.

This method is based on the fact that the square matrix A can be factorized into the form LU where L is unit lower triangular matrix and U is upper triangular matrix, if all the principal minors of A are non-singular, i.e., if

$$a_{11} \neq 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0, \text{ etc.}$$

We shall explain this method by taking a set of three equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \longrightarrow (1)$$

(1) can be written in the form

$$AX = B \longrightarrow (2)$$

Let $\quad A = LU \longrightarrow (3)$

where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \text{ and } U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

Writing $A = LU$ in (2), we have

$$LUX = B \longrightarrow (4)$$

Putting $UX = Y$, the eqn. (4) becomes

$$LY = B \longrightarrow (5)$$

where, $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, say.

Now, (5) $\Rightarrow$

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

This is equivalent to the system

$$\left. \begin{aligned} y_1 &= b_1 \\ l_{21} y_1 + y_2 &= b_2 \\ l_{31} y_1 + l_{32} y_2 + y_3 &= b_3 \end{aligned} \right\} \longrightarrow (6)$$

The values of y_1, y_2, y_3 are obtained by forward substitution.

When we the system $UX = Y$ gives:

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} u_{11}x_1 + u_{12}x_2 + u_{13}x_3 &= y_1 \\ u_{22}x_2 + u_{23}x_3 &= y_2 \\ u_{33}x_3 &= y_3 \end{aligned}\right\} \rightarrow (7)$$

By the backward substitution, we get the values of x_1, x_2 and x_3. Then solⁿ is

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Procedure of computing L and U :

From the relation $A = LU$, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}a_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Equating the corresponding elts. on both sides, we get

$u_{11} = a_{11}$; $u_{12} = a_{12}$; $u_{13} = a_{13}$

$l_{21}u_{11} = a_{21} \Rightarrow l_{21} = a_{21}/a_{11}$,

$l_{21}u_{12} + u_{22} = a_{22} \Rightarrow u_{22} = a_{22} - (a_{21}/a_{11})a_{12}$,

$l_{21}u_{13} + u_{23} = a_{23} \Rightarrow u_{23} = a_{23} - (a_{21}/a_{11})a_{13}$,

$l_{31}a_{11} = a_{31} \Rightarrow l_{31} = a_{31}/a_{11}$,

$l_{31}u_{12} + l_{32}u_{22} = a_{32} \Rightarrow l_{32} = \dfrac{a_{32} - (a_{31}/a_{11})a_{12}}{u_{22}}$,

and $l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33}$ which gives u_{33}.

i) 1st row
ii) 1st colᵐ
iii) 2nd row
(iii) 2nd colᵐ.
(v) 3rd row.

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