Lake Pollution Model:

Lake pollution model is concerned with the pollution in the lake and rivers. It has become a major problem particularly over the last few years. This model can be considered as a compartmental model with a single compartment, the lake. It can be represented in the form of the following compartmental diagram.

$$\xrightarrow{\text{mass inflow}} \boxed{\begin{array}{c}\text{mass of pollutant} \\ \text{in lake}\end{array}} \xrightarrow{\text{mass outflow}}$$

Fig. 23: Input-output compartmental diagram of lake pollution.

By balance law, the word equation for the mass of pollution in lake

$$\left\{\begin{array}{c}\text{rate of change} \\ \text{of mass} \\ \text{in lake}\end{array}\right\} = \left\{\begin{array}{c}\text{rate of} \\ \text{mass} \\ \text{enters lake}\end{array}\right\} - \left\{\begin{array}{c}\text{rate of} \\ \text{mass} \\ \text{leave lake}\end{array}\right\}$$

Formulation of differential Eqn. (Mathematical model):

We assume that the volume of the lake is constant, and that it is continuously well mixed so that the pollution is uniform throughout.

Let,

V = volume of the lake,

$C(t)$ = concentration of the pollutant in the lake at time t,

f = rate at which water flows out of the lake in m^3/day.

$m(t)$ = mass of pollutant of the lake in time t.

Since the volume is constant, we have

$c = \frac{m}{V}$

$$\left\{\begin{matrix}\text{flow of}\\ \text{mixture}\\ \text{into lake}\end{matrix}\right\} = \left\{\begin{matrix}\text{flow of}\\ \text{mixture}\\ \text{out of lake}\end{matrix}\right\} = f$$

Now, applying the balance law to the mass of the pollutant $m(t)$, we get the word eqn

$$\left\{\begin{matrix}\text{rate of change of}\\ \text{mass of pollutant}\\ \text{in lake}\end{matrix}\right\} = \left\{\begin{matrix}\text{rate at which}\\ \text{the pollutant}\\ \text{enters the lake}\end{matrix}\right\} - \left\{\begin{matrix}\text{rate at which}\\ \text{the pollutant}\\ \text{leaves the lake}\end{matrix}\right\}$$

$$\Rightarrow \quad \frac{d\, m(t)}{dt} = f c_{in} - f \frac{m(t)}{V} \quad \longrightarrow (1)$$

where c_{in} is the concentration (in units of mass per unit of volume, such as g/m^3) of the pollutant in the flow entering the lake.

Since $m(t) = c(t)\, V$

$\therefore\ m'(t) = c'(t)\, V \quad (\because$ volume is const.$)$

$\Rightarrow c'(t) = \frac{m'(t)}{V}$

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putting this in (1),

$$V\,c'(t) = f\,c_{in} - f\,c(t)$$

$$\Rightarrow c'(t) = \frac{f}{V} c_{in} - \frac{f}{V} c(t)$$

$$\Rightarrow \boxed{\frac{dc}{dt} = \frac{f}{V}(c_{in} - c)} \longrightarrow (2)$$

which is the differential eqn. of lake pollution model.

Note: Concentration = mass of pollutant/unit volume.

Ex. Solve the differential eqn

$$\frac{dc}{dt} = \frac{f}{V}(c_{in} - c)$$

with the initial cantion $c(0) = c_0$

Solⁿ Given eqn is

$$\frac{dc}{dt} = \frac{f}{V}(c_{in} - c) \longrightarrow (1)$$

or, $$\frac{dc}{c_{in} - c} = \frac{f}{V}\,dt$$

Integrating

$$\int \frac{dc}{c_{in} - c} = \int \frac{f}{V}\,dt$$

or, $$-\log|c_{in} - c| = \frac{f}{V}t + A$$, A → const. of Integration

or, $$\log|c_{in} - c| = -\frac{f}{V}t - A$$

or, $$c_{in} - c = e^{-\frac{f}{V}t - A} = e^{-A}\cdot e^{-\frac{f}{V}t}$$

or, $$c = c_{in} - e^{-A} e^{-\frac{f}{V}t}$$

or, $$\boxed{c(t) = c_{in} - B e^{-\frac{f}{V}t}} \longrightarrow (2)$$, where $e^{-A} = B$ which is the G.S. of (1).

Now, using initial condition $c(0) = c_0$, we get

$$c_0 = c_{in} - B e^0$$

$$\Rightarrow B = c_{in} - c_0$$

Putting this value of B in (2), we get

$$\boxed{c(t) = c_{in} - (c_{in} - c_0)\, e^{-\frac{f}{V}t}} \longrightarrow (3)$$

which is the required solⁿ.

Ex. How long will it take for a lake's pollution level to reach 5% of its initial level, if only fresh water flow into the lake.

Soln. We have, with usual notations,

$$c(t) = c_{in} - (c_{in} - c_0)\, e^{-\frac{f}{V}t} \longrightarrow (1)$$

$\because$ only fresh water flow into the lake,
(no pollutant)

$\therefore \quad c_{in} = 0.$

$$\therefore (1) \Rightarrow c(t) = c_0 e^{-\frac{f}{V}t}$$

$$\Rightarrow \log_e c(t) = \log_e c_0 - \frac{f}{V}t$$

$$\Rightarrow \frac{ft}{V} = \log_e\left(\frac{c_0}{c(t)}\right)$$

$$\Rightarrow t = \frac{V}{f}\log_e\left\{\frac{c_0}{c(t)}\right\} \longrightarrow (2)$$

Given,

$$c(t) = 5\% \text{ of } C_0$$
$$= \frac{5}{100} C_0$$
$$= 0.05\, C_0$$

Putting this value of $c(t)$ in eqn (2), we get

$$t = \frac{V}{f} \log_e \frac{C_0}{0.05\, C_0}$$
$$= \frac{V}{f} \log_e (20)$$
$$= \frac{V}{f} (2.9957)$$
$$\simeq \frac{V}{f} \times 3$$
$$\Rightarrow \boxed{t \simeq \frac{3V}{f} \text{ days}}$$

unit of t:

$V \to m^3$

$f \to m^3/day$

$\therefore t \to \frac{m^3}{m^3/day} \equiv day$

Example :- Let in a lake the polution level is 7%. If the concentration of the incoming water is 2% and 10,000 litres per day water is allowed to enter the lake, find time when pollution level is 5% and volume of the lake is 2,00,000 litres. Also find pollution after 32 days.

<u>Solution</u> :→ We have, with usual notations,

$$c(t) = c_{in} - (c_{in} - c_0)\, e^{-\frac{f}{V}t} \longrightarrow (1).$$

Here given,

$V = 200,000$

$f = 10,000$

$c_{in} = 0.02$

$c(t) = 0.05$

$c_0 = 0.07$

Putting these values in (1), we get

$$0.05 = 0.02 - (0.02 - 0.07)\, e^{-\frac{10,000}{200,000}t}$$

$$\Rightarrow 0.02 - 0.05 = (0.02 - 0.07)\, e^{\frac{-1}{20}t}$$

$$\Rightarrow e^{\frac{-t}{20}} = \frac{-0.03}{-0.05} = \frac{3}{5} = 0.6$$

Taking logarithm on both sides,

$$-\frac{t}{20} = \log_e(0.6)$$

$$\Rightarrow -\frac{t}{20} = -0.51082$$

$$\Rightarrow t = 10.2164 \text{ days.}$$

Hence the required time = 10.2164 days

<u>2nd Part</u>: Putting $t = 32$ in eqn. (1), we get

$$c(t) = c_{in} - (c_{in} - c_0)\, e^{-\frac{f}{V}32}$$

$$= 0.02 - (0.02 - 0.07)\, e^{-\frac{10000}{200,000} \times 32}$$

$$= 0.02 + 0.05\, e^{-1.6} = 0.02 + 0.05(0.201896) \cong 0.30$$

∴ Pollution level after 32 days = 3% (approx)

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