

Lake Pollution Model :

Lake pollution model is concerned with the pollution in the lake and rivers. It has become a major problem particularly over the last few years. This model can be considered as a compartmental model with a single compartment, the lake. It can be represented in the form of the following compartmental diagram.

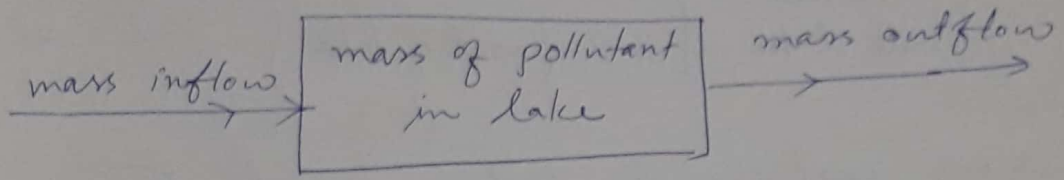


Fig. 23: Input-output compartmental diagram of lake pollution.

By balance law, the word equation for the mass of pollution in lake

$$\left\{ \begin{array}{l} \text{rate of change} \\ \text{of mass} \\ \text{in lake} \end{array} \right\} = \left\{ \begin{array}{l} \text{rate of} \\ \text{mass} \\ \text{enters lake} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate of} \\ \text{mass} \\ \text{leave lake} \end{array} \right\}$$

Formulation of differential Eqn. (Mathematical model) :

We assume that the volume of the lake is constant, and that it is continuously well mixed so that the pollution is uniform throughout.

Let,

$V =$ volume of the lake,

$C(t) =$ concentration of the pollutant in the lake at time t ,

$f =$ rate at which water flows out of the lake in m^3/day ,

$m(t) =$ mass of pollutant of the lake in time t .

Since the volume is constant, we have $c = \frac{m}{V}$

$$\left\{ \begin{array}{l} \text{flow of} \\ \text{mixture} \\ \text{into lake} \end{array} \right\} = \left\{ \begin{array}{l} \text{flow of} \\ \text{mixture} \\ \text{out of lake} \end{array} \right\} = f$$

Now, applying the balance law to the mass of the pollutant $m(t)$, we get

from the word eqn

$$\left\{ \begin{array}{l} \text{rate of change of} \\ \text{mass of pollutant} \\ \text{in lake} \end{array} \right\} = \left\{ \begin{array}{l} \text{rate at which} \\ \text{the pollutant} \\ \text{enters the lake} \end{array} \right\} - \left\{ \begin{array}{l} \text{rate at which} \\ \text{the pollutant} \\ \text{leaves the lake} \end{array} \right\}$$

$$\Rightarrow \frac{d}{dt} m(t) = f c_{in} - f \frac{m(t)}{V} \quad \text{--- (1)}$$

where c_{in} is the concentration (in units of mass per unit of volume, such as g/m^3) of the pollutant in the flow entering the lake.

since

$$m(t) = c(t) V$$

$$\therefore m'(t) = c'(t) V \quad (\because \text{volume is const.})$$

$$\Rightarrow c'(t) = \frac{m'(t)}{V}$$

putting this in (1),

$$V c'(t) = f c_{in} - f c(t)$$

$$\Rightarrow c'(t) = \frac{f}{V} c_{in} - \frac{f}{V} c(t)$$

$$\Rightarrow \boxed{\frac{dc}{dt} = \frac{f}{V} (c_{in} - c)} \longrightarrow (2)$$

which is the differential eqn. of lake pollution model.

Note: Concentration = mass of pollutant/unit volume

Ex. Solve the differential eqn

$$\frac{dc}{dt} = \frac{f}{V} (c_{in} - c)$$

with the initial condition $c(0) = c_0$

Solⁿ Given eqn is

$$\frac{dc}{dt} = \frac{f}{V} (c_{in} - c) \longrightarrow (1)$$

$$\text{or, } \frac{dc}{c_{in} - c} = \frac{f}{V} dt$$

Integrating

$$\int \frac{dc}{c_{in} - c} = \int \frac{f}{V} dt$$

$$\text{or, } -\log|c_{in} - c| = \frac{f}{V} t + A \quad , \quad A \rightarrow \text{const. of Integration}$$

$$\text{or, } \log|c_{in} - c| = -\frac{f}{V} t - A$$

$$\text{or, } c_{in} - c = e^{-\frac{f}{V} t - A} = e^{-A} \cdot e^{-\frac{f}{V} t}$$

$$\text{or, } c = c_{in} - e^{-A} e^{-\frac{f}{V} t}$$

$$\text{or, } \boxed{c(t) = c_{in} - B e^{-\frac{f}{V} t}} \longrightarrow (2), \text{ where } e^{-A} = B$$

which is the G.S of (1).

Now, using initial condition $c(0) = c_0$, we get

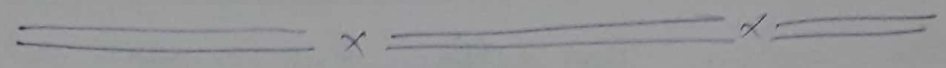
$$c_0 = c_{in} - B e^0$$

$$\Rightarrow B = c_{in} - c_0$$

Putting this value of B in (2), we get

$$c(t) = c_{in} - (c_{in} - c_0) e^{-\frac{f}{V} t} \rightarrow (3)$$

which is the required solⁿ.



Ex. How long will it take for a lake's pollution level to reach 5% of its initial level, if only fresh water flow into the lake.

Soln. We have, with usual notations,

$$c(t) = c_{in} - (c_{in} - c_0) e^{-\frac{f}{V} t} \rightarrow (1)$$

\therefore only fresh water flow into the lake,
(no pollutant)

$$\therefore c_{in} = 0.$$

$$\therefore (1) \Rightarrow c(t) = c_0 e^{-\frac{f}{V} t}$$

$$\Rightarrow \log_e c(t) = \log_e c_0 - \frac{f}{V} t$$

$$\Rightarrow \frac{f t}{V} = \log_e \left(\frac{c_0}{c(t)} \right)$$

$$\Rightarrow t = \frac{V}{f} \log_e \left\{ \frac{c_0}{c(t)} \right\} \rightarrow (2)$$

Given,

$$\begin{aligned}
 c(t) &= 5\% \text{ of } c_0 \\
 &= \frac{5}{100} c_0 \\
 &= 0.05 c_0
 \end{aligned}$$

Putting this value of $c(t)$ in eqn (2), we get

$$t = \frac{V}{f} \log_e \frac{c_0}{0.05 c_0}$$

$$= \frac{V}{f} \log_e (20)$$

$$= \frac{V}{f} (2.9957)$$

$$\approx \frac{V}{f} \times 3$$

$$\Rightarrow \boxed{t \approx \frac{3V}{f} \text{ days}}$$

unit of t:

$$V \rightarrow m^3$$

$$f \rightarrow m^3/\text{day}$$

$$\therefore t \rightarrow \frac{m^3}{m^3/\text{day}} = \text{day}$$

Example :- Let in a lake the pollution level is 4%. If the concentration of the incoming water is 2% and 10,000 litres per day water is allowed to enter the lake, find time when pollution level is 5% and volume of the lake is 2,00,000 litres. Also find pollution after 32 days.

Solution \Rightarrow We have, with usual notations,

$$c(t) = c_{in} - (c_{in} - c_0) e^{-\frac{f}{V}t} \longrightarrow (1)$$

Here given,

$$V = 200,000$$

$$f = 10,000$$

$$c_{in} = 0.02$$

$$c(t) = 0.05$$

$$c_0 = 0.07$$

Putting these values in (1), we get

$$0.05 = 0.02 - (0.02 - 0.07) e^{-\frac{10,000}{200,000}t}$$

$$\Rightarrow 0.02 - 0.05 = (0.02 - 0.07) e^{-\frac{t}{20}}$$

$$\Rightarrow e^{-\frac{t}{20}} = \frac{-0.03}{-0.05} = \frac{3}{5} = 0.6$$

Taking logarithm on both sides,

$$-\frac{t}{20} = \log_e(0.6)$$

$$\Rightarrow -\frac{t}{20} = -0.51082$$

$$\Rightarrow t = 10.2164 \text{ days.}$$

Hence the required time = 10.2164 days

2nd Part: Putting $t = 32$ in eqn. (1), we get

$$c(t) = c_{in} - (c_{in} - c_0) e^{-\frac{f}{V}t}$$

$$= 0.02 - (0.02 - 0.07) e^{-\frac{10000}{200,000} \times 32}$$

$$= 0.02 + 0.05 e^{-1.6} = 0.02 + 0.05(0.201896) \cong 0.30$$

\therefore Pollution level after 32 days = 3% (approx).