(5) Newton Raphson Method :
(or, Newton's method)

Let (x_0) be an approximate value of a root of the equation $f(x) = 0$, and let $x_0 + h$ be the exact value of the corresponding root, where h is very small quantity.

$\because$ $x_0 + h$ is a root of the eqn $f(x) = 0$,

$$\therefore \quad f(x_0 + h) = 0 \longrightarrow (1)$$

Expanding (1) by Taylor's theorem, we get-

$$f(x_0 + h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \cdots\cdots = 0$$

Since h is very small, neglecting second and higher order terms and taking the first approximation, we have

$$f(x_0) + h f'(x_0) = 0$$

$$\Rightarrow h = -\frac{f(x_0)}{f'(x_0)}, \quad \text{provided } f'(x_0) \neq 0.$$

$\therefore$ first approx. of the root x_1 is given by

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)} \longrightarrow (2)$$

Relation (2) gives the improved value of the root over the previous one. Now substituting x_1 for x_0 and x_2 for x_1 in (2), we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \longrightarrow (3)$$

In general, we get the iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \longrightarrow (4).$$

Formula (4) is known as Newton-Raphson method.

Note: Newton Raphson method is applicable only when h is very small, i.e. when $f'(x)$ is large.

Q. Obtain $\sqrt{(12)}$ to five places of decimal by Newton - Raphson method.

Soln. Let us take $x = \sqrt{12}$, so that

$$x^2 - 12 = 0.$$

It is obvious that x^2-12 becomes negative for $x=3$, whereas it is positive for $x = 3.5$. Hence the root of $x^2-12=0$ lies in the interval $(3, 3.5)$.

Now, let $f(x) = x^2-12 = 0 \longrightarrow (1)$

$$\therefore f'(x) = 2x.$$

Now, by Newton-Raphson Method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \longrightarrow (2)$$

$\underline{\underline{n=1}}$ $\Rightarrow$ $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$$= 3.5 - \frac{f(3.5)}{f'(3.5)}, \text{ taking } x_0 = 3.5.$$

$$= 3.5 - \frac{(3.5)^2-12}{2\times(3.5)}$$

$$= 3.5 - \frac{0.25}{7} = 3.5 - 0.03571$$

$$= 3.464$$

Again we have

$$x_2 = x_1 - \frac{f(x_1) - 12}{f'(x_1)}$$

$$= x_1 - \frac{x_1^2 - 12}{2x_1}$$

$$= 3.464 - \frac{(3.464)^2 - 12}{2(3.464)}$$

$$= 3.464 - \frac{(-0.000704)}{6.928}$$

$$= 3.464 + 0.0001016166$$

$$= 3.4641$$

Also,

$$x_3 = x_2 - \frac{f(x_2) - 12}{f'(x_2)}$$

$$= 3.4641 - \frac{(3.4641)^2 - 12}{2\times(3.4641)}$$

$$= 3.4641 - \frac{(-0.00001119)}{6.9282}$$

$$= 3.4641 + 0.0000016151$$

$$= 3.4641016151$$

$$= 3.4641$$

Hence the required root is 3.4641.

i.e. $\sqrt{12} = 3.4641$ ← Ans.