# Compartmental Models (C.M.)

Compartmental model is a model in which there is a place called "compartment" which has amount of substance in and out over time.

OR

Models of Processes (natural or artificial) which have inputs to and outputs from a "compartment" over time, are called C.M.

One example of compartmental notion is the amount of carbondioxide ($CO_2$) in the earth atmosphere. Here the compartment is the atmosphere. The input of $CO_2$ occurs through many processes, such as burning, and the output of $CO_2$ occurs through processes such as plant's photosynthesis. This phenomena is illustrated in the following diagram known as "compartmental diagram".

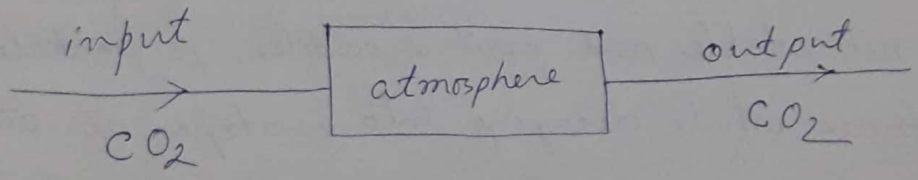


Fig: 2.1 : Input-output Compartmental diagram for $CO_2$.

## Balance law :→

Balance law states that the rate of change of amount of a substance ($CO_2$ in the above example) is equal to the 'rate in' minus 'rate out' of the compartment.

i.e.
$$\left\{\begin{matrix}\text{net rate of change} \\ \text{of a substance}\end{matrix}\right\} = \left\{\begin{matrix}\text{rate} \\ \text{in}\end{matrix}\right\} - \left\{\begin{matrix}\text{rate} \\ \text{out}\end{matrix}\right\}$$

This is known as word equation of the model.

With the help of this balance law, we can formulate some compartmental models in Mathematical terms (i.e. mathematical model). Some examples of processes, which can be describe by this law, are —

(i) the decay process of radioactive elements,

(ii) births and deaths in a population.

(iii) pollution into and out of a lake or river, or the atmosphere;

(iv) drug assimilation into, and removal from, the bloodstream.

## Exponential Decay Model (of radiactive material)

Radioactive elements are those elements which are not stable and emit $\alpha$-particles, $\beta$-particles or photones while decaying into isotopes of other elements. Exponential decay model for radioactive decay can be considered as a compartmental model with compartment being the radioactive material with no input but output as decay of radioactive sample over time. Here the compartmental diagram is :

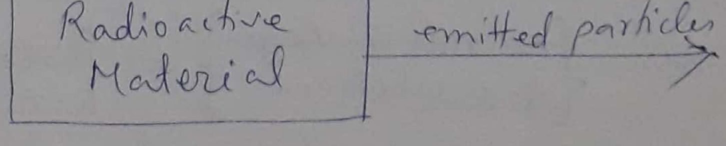


Fig 2.2 : Input-output compartmental diagram for radioactive nuclei

By Balance law, word equation is

$$\left\{\begin{array}{c}\text{rate of change of} \\ \text{radioactive material} \\ \text{at time } t\end{array}\right\} = -\left\{\begin{array}{c}\text{rate of amount} \\ \text{of radioactive} \\ \text{material decayed}\end{array}\right\} \longrightarrow (1)$$

We make the following assumptions:

(i) We assume that the amount of an element present is large enough so that we are justified in ignoring random fluctuations.

(ii) We assume the process is continuous in time.

(iii) We assume a fixed rate of decay for an elt.

(iv) We assume there is no increase in mass of the body of material.

Formulation of the Differential Equation (i.e. mathematical model)

Let, $N(t)$ = number of radioactive nuclei at time $t$.

and $\Delta t$ = small change in time.

We know that the change in nuclei is proportional to the number of nuclei at the start of the time period,

i.e, $\frac{dN(t)}{dt} \propto N(t)$

$\therefore$ $\boxed{\frac{dN(t)}{dt} = -kN(t)} \longrightarrow (2)$

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where k is a positive constant of proportionality indicating the rate of decay per nucleus in unit time.

Eqn. (2) is the required exponential decay model.

If at initial condition, i.e., when $t=0$, the number of radioactive nuclei is $n_0$, i.e. $N(0)=n_0$, then we have an initial value problem (IVP) corresponding to the exponential decay model as

$$\boxed{\frac{dN(t)}{dt} = -kN(t)\ ;\ N(0)=n_0} \longrightarrow (3).$$

Note: The value of decay constant k depends on the particular radioactive isotopes. For $^{14}C$, it is known that $k = 0.0001216$ if $t$ is measured in years.

Example 1. Solve the exponential decay model

$$\frac{dN(t)}{dt} = -kN(t)$$

with the initial condition $N(t_0) = n_0$.

Solution:→ We have the IVP

$$\frac{dN(t)}{dt} = -kN(t)$$

$$\Rightarrow \quad \frac{dN}{N} = -k\,dt \quad \text{(variables are separated)}$$

Integrating

$$\int \frac{dN}{N} = \int (-k\,dt)$$

$$\Rightarrow \log_e N = -kt + \log c \text{, where c is const. of integration.}$$

$$\Rightarrow \frac{N}{c} = e^{-kt}$$

$$\Rightarrow N(t) = c\,e^{-kt} \longrightarrow (1)$$

Initially, given, $N(t_0) = n_0$.

$\therefore (1) \Rightarrow n_0 = C e^{-kt_0}$

$\Rightarrow C = n_0 e^{kt_0}$

Putting this value of C in (1), we get

$$N(t) = n_0 e^{kt_0} e^{-kt}.$$

$$\Rightarrow \boxed{N(t) = n_0 e^{-k(t-t_0)}}$$

which is the required solution.

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<u>Example 2.</u> Solve the I.V.P

$$\frac{dN(t)}{dt} = -k\, N(t), \quad N(0) = n_0$$

on the interval $[0, t]$.

<u>Soln.</u> We have the IVP

$$\frac{dN(t)}{dt} = -k\, N(t)$$

$\Rightarrow \dfrac{dN(t)}{N(t)} = -k\,dt$, variables are separated.

$\Rightarrow \displaystyle\int_{n_0}^{N} \frac{dN}{N} = -\int_0^t k\,dt$, integrating over $[0, t]$ $\left[\begin{array}{l}\text{when } t=0,\ N=n_0 \\ \quad ,, \quad t=t,\ N=N\end{array}\right]$

$\Rightarrow \left[\log_e N\right]_{n_0}^{N} = \left[-kt\right]_0^t$

$\Rightarrow \log_e N - \log_e n_0 = -kt + 0$

$\Rightarrow \log_e \dfrac{N}{n_0} = -kt$

$\Rightarrow \dfrac{N}{n_0} = e^{-kt}$

$\Rightarrow N = n_0 e^{-kt}$

which is the solution of the IVP.