(4) Iteration Method (OR, Fixed point iteration method):

Let the equation be $f(x) = 0 \longrightarrow (1)$.

This method is used when we can find x from (1) as

$$x = \phi(x) \longrightarrow (2)$$

If $\left| \phi'(x) \right|_{\text{at } x = x_0} < 1$, then iterative

method is applied and the successive approximation is given by (i.e. iterative formula is given by)

$$\boxed{x_n = \phi(x_{n-1}), \quad n = 1, 2, 3, \cdots}$$

i.e. $x_1 = \phi(x_0)$
$x_2 = \phi(x_1)$
$x_3 = \phi(x_2)$
$\vdots$

<u>Note</u>:- If $\left| \phi'(x) \right|_{x = x_0} \geq 1$, then this method cannot be applied.

<u>Theorem</u>:→ Let $x = \alpha$ be a root of $f(x) = 0$ and let I be an interval containing the point $x = \alpha$. Let $\phi(x)$ and $\phi'(x)$ be continuous in I where $\phi(x)$ is defined by the equation $x = \phi(x)$ which is equivalent to $f(x) = 0$. Then if $|f'(x)| < 1$ for all x in I, the sequence of approximations $x_0, x_1, x_2, \cdots, x_n$ defined by $x_n = \phi(x_{n-1})$ converges to the root α, provided that the initial approximation $x_0 \in I$.

Ex. Find a real root of the equation

$$f(x) = x^3 + x^2 - 1 = 0,$$

by using iteration method.

Solⁿ Given eqn is

$$f(x) = x^3 + x^2 - 1 = 0 \longrightarrow (1)$$

Here,

$$f(0) = -1 < 0$$
$$f(1) = 1^3 - 1^2 - 1 = 1 > 0$$

Since $f(0) < 0 < f(1)$, hence the root lies in between 0 and 1.

Now, we have from (1),

$$x^3 + x^2 - 1 = 0$$

$$\Rightarrow x^2(x+1) = 1$$

$$\Rightarrow x^2 = \frac{1}{x+1}$$

$$\Rightarrow x = \frac{1}{\sqrt{x+1}}$$

$$\Rightarrow x = \phi(x), \text{ say}$$

where, $\phi(x) = \dfrac{1}{\sqrt{1+x}}$.

Now,

$$\phi'(x) = \frac{-1}{2(1+x)^{3/2}}$$

Again,

$$|\phi'(x)| = \left|\frac{1}{2(x+1)^{3/2}}\right| < 1 \text{ for } x < 1$$

Hence iterative method is applicable.

Starting with $x_0 = 0.5$, we get

$$x_1 = \phi(x_0)$$

$$= \frac{1}{\sqrt{1+x_0}} = \frac{1}{\sqrt{1+0.5}} = \frac{1}{\sqrt{1.5}} = 0.81649$$

$$x_2 = \phi(x_1)$$

or, $$x_2 = \frac{1}{\sqrt{1+x_1}} = \frac{1}{\sqrt{1.81649}} = 0.74196$$

$$x_3 = \phi(x_2) = \frac{1}{\sqrt{1+x_2}} = \frac{1}{\sqrt{1.74196}} = 0.75767$$

$$x_4 = \phi(x_3) = \frac{1}{\sqrt{1.75767}} = 0.75427$$

$$x_5 = \phi(x_4) = \frac{1}{\sqrt{1.75427}} \simeq 0.75500$$

$$x_6 = \phi(x_5) = \frac{1}{\sqrt{1.75500}} = 0.75485$$

$$x_7 = \phi(x_6) = \frac{1}{\sqrt{1.75485}} = 0.75488$$

$$x_8 = \phi(x_7) = \frac{1}{\sqrt{1.75488}} = 0.75487$$

$$x_9 = \phi(x_8) = \frac{1}{\sqrt{1.75487}} = 0.75487, \text{ etc.}$$

H.W. Ex. Find the roots of $\cos x = 3x - 1$, correct to four decimal places by using iterative method.