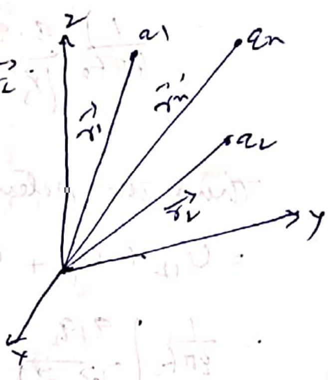
C.A. 5 "Electrostatic Energy":

Potential energy of charge:- Electric potential at a point in an electric field is defined as the work done in taking a unit positive charge from infinity to that point against electrical forces.

If V is the electrical potential at a point, then work done in bringing a test charge q_0 from infinity to that point $= q_0 V$. This work is stored in the test charge as its potential energy.

P.E of a system of charge:- The potential energy of a systems of charges is defined as the minimum ~~energy~~ external work done in assembling the system.

Let we have a set of charges q_1 at $\vec{r_1}$, q_2 at $\vec{r_2}$ ... q_n at $\vec{r_n}$



To begin with all the charges are at infinity and are also at infinite distance from one another so that there is no interaction between them. At this stage the configuration has zero potential energy.

Now we bring the charge q_1 from infinity to $\vec{r_1}$ infinitely slowly keeping the other charges at ~~infinity~~ $q_2\, q_3 \cdots q_n$ still at infinity. Since all other charges are at infinity, the electric potential already existing at the position $\vec{r_1}$ is zero. Hence no work is done in bringing the charge q_1 from infinity to the position $\vec{r_1}$.

P.E of system of two charges:- Now keeping the charge q_1 fixed at $\vec{r_1}$ and bring the charge q_2 from infinity to $\vec{r_2}$ keeping the charge $q_3\, q_4 \cdots q_n$ still at infinity.

The potential at $\vec{r_2}$ due to the charge q_1 at $\vec{r_1}$ is given by $\frac{1}{4\pi\epsilon_0}\frac{q_1}{|\vec{r_2}-\vec{r_1}|}$

$\longrightarrow$

Hence work done in bringing the charge q_2 from infinity to $\vec{r_2}$ $= U_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{|\vec{r_2}-\vec{r_1}|} q_2$

This gives the potential energy of a system of two charges.

P.E of a system of three charges :–

Now keep the charges q_1 fixed at $\vec{r_1}$ and q_2 fixed at $\vec{r_2}$ and bring the charge q_3 from infinity to $\vec{r_3}$. The electric potential at $\vec{r_3}$ due to the charge q_1 and q_2 at $\vec{r_1}$ and $\vec{r_2}$ respectively is given by

$$\frac{1}{4\pi\epsilon_0}\frac{q_1}{|\vec{r_3}-\vec{r_1}|} + \frac{1}{4\pi\epsilon_0}\frac{q_2}{|\vec{r_3}-\vec{r_2}|}$$

∴ The work done in bringing the charges q_3 from infinity to $\vec{r_3}$

$$\frac{1}{4\pi\epsilon_0}\frac{q_1 q_3}{|\vec{r_3}-\vec{r_1}|} + \frac{1}{4\pi\epsilon_0}\frac{q_2 q_3}{|\vec{r_3}-\vec{r_2}|} = U_{13} + U_{23}$$

Thus the potential energy of a system of three charges

$= U_{12} + U_{13} + U_{23}$.

$= \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 q_2}{|\vec{r_2}-\vec{r_1}|} + \frac{q_1 q_3}{|\vec{r_3}-\vec{r_1}|} + \frac{q_2 q_3}{|\vec{r_3}-\vec{r_2}|}\right]$ which is equal to the sum of the potential energy for a system of ~~four charges~~ various pairs.

P.E. of a system of n charges :–

proceeding as above the potential energy for a system of four charges $U = U_{12} + U_{13} + U_{14} + U_{23} + U_{24} + U_{34}$

∴ For a system of n charges, we have

$$U = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_i q_j}{|\vec{r_i}-\vec{r_j}|} \quad \text{---- (1)} \quad (i \neq j)$$

The factor $\frac{1}{2}$ has been introduced because in the above summation we get term U_{12} and another term U_{21}. The two are equal. But in calculation of potential energy

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We have to ~~acco~~ count each pair only once.

Relation (1) can be put in the form

$$U = \frac{1}{2}\sum_{i=1}^{n} q_i \sum_{j=1}^{n} \frac{q_j}{|\vec{r_i}-\vec{r_j}|}$$

Now $\sum_{j=1}^{n} \frac{q_j}{|\vec{r_i}-\vec{r_j}|} = V_i$, the electric potential at the charge q_i due to all other charges.

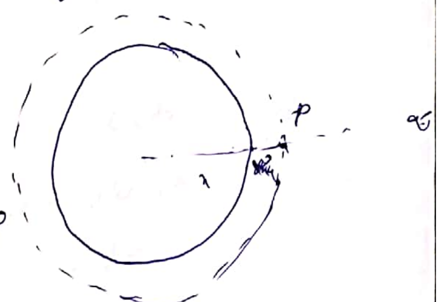
Hence $U = \frac{1}{2}\sum_{i=1}^{n} q_i V_i$

C.D.C. Potential energy of a charged sphere :-

Suppose the charge is assembled bit by bit by bringing small instalments of charge dq each from infinity. Let x be the radius of sphere of charge at any instant and ρ its volume charge density. Then

potential of the sphere of charge at a point on its surface $= \frac{1}{4\pi\epsilon_0}\frac{\frac{4}{3}\pi x^3\rho}{x}$, where $\frac{4}{3}\pi x^3$ gives the volume of the sphere of radius x and $\frac{4}{3}\pi x^3\rho$ the total charge on the assembled sphere.

If we bring a further charge dq from infinity so that the radius of the sphere increases from x to $x+dx$, then the additional charge is contained in the shell of radius x and thickness dx (having charge density ρ), then $dq = 4\pi x^2 dx \rho$.



Work done in bringing this charge dq

$$dW = \frac{1}{4\pi\epsilon_0}\frac{4}{3}\pi x^3\rho\, 4\pi x^2 dx \rho$$

$$= \frac{1}{4\pi\epsilon_0}\frac{16}{3}\pi^2 x^4 \rho^2 dx.$$

$\therefore$ Total work done in assembling the ~~charge~~ sphere of charge of radius R

$\rightarrow$

$$W = \int dW = \frac{1}{4\pi\epsilon_0}\frac{16}{3}\pi^2\rho^2\int_0^R r^4\,dr.$$

$$= \frac{1}{4\pi\epsilon_0}\frac{16}{3}\pi^2\rho^2\left[\frac{r^5}{5}\right]_0^R$$

$$= \frac{1}{4\pi\epsilon_0}\frac{16}{3}\pi^2\rho^2\frac{R^5}{5}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{16}{15}\pi^2\rho^2R^5$$

But $\frac{4}{3}\pi R^3\rho = q$ the total charge in the sphere

$$\therefore q^2 = \frac{16}{9}\pi^2R^6\rho^2 \text{ and } \frac{16}{15}\pi^2\rho^2R^5 = \frac{3}{5}\frac{q^2}{R}$$

Hence $W = \frac{1}{4\pi\epsilon_0}\frac{3}{5}\frac{q^2}{R}$

This work is stored in the sphere as its electrical potential energy. Hence

$$U = \frac{1}{4\pi\epsilon_0}\frac{3}{5}\frac{q^2}{R}.$$