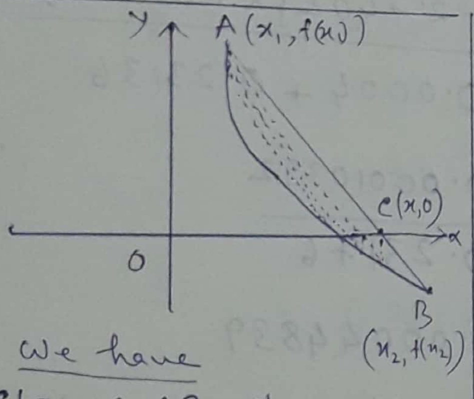
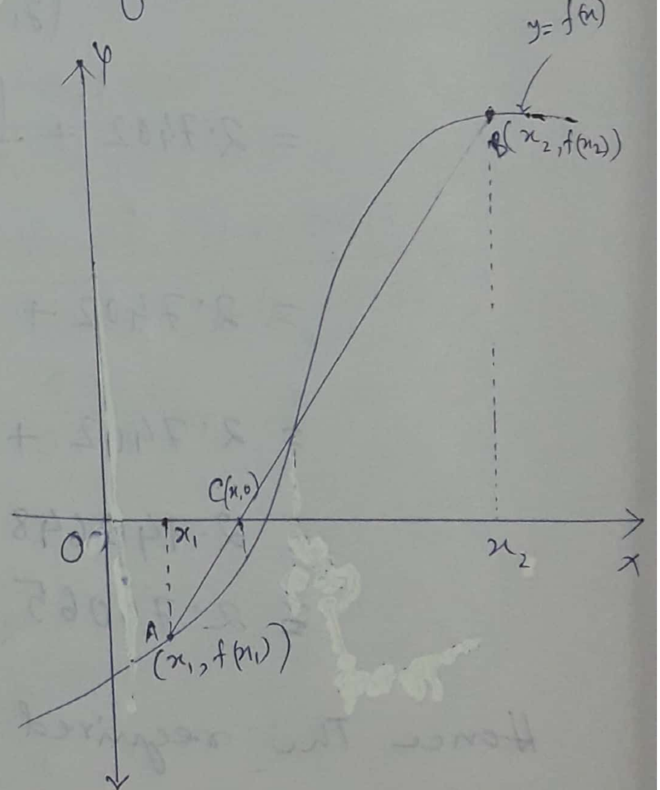
(3) The secant Method :

[This method is quite similar to that of Regula-falsi method except for the condition $f(x_1) \cdot f(x_2) < 0$.

Here the graph of the function $y = f(x)$ in the neighbourhood of the root is approximated by a secant line (i.e. chord). Further here the interval at each iteration may not contain the root.]

Let the equation be $y = f(x) = 0$, real root of which is to be find out.





We have

Slope of AB = slope of AC

$$\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{0 - f(x_1)}{x - x_1}$$

$$\Rightarrow x - x_1 = \frac{(x_2 - x_1)}{f(x_2) - f(x_1)}(-f(x_1))$$

$$\Rightarrow x = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

writing x_3 for x, $x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

$\therefore$ Gen formula: $x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$

We consider a small interval (x_1, x_2) in the domain of the function $y = f(x)$.
[Here, the root is not necessarily lie in the interval (x_1, x_2).]

We construct a line (secant) through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$, as the shown in the above figure. In slope-intercept form, the eqn. of this line is

$$y - f(x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_2) \rightarrow (1)$$

(x_1, y_1) A, (x, y), (x_2, y_2) B

Eqn. of line AB

$$y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2)$$

OR

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

The root of this linear function, that is the value of x such that $y = 0$ is given by

$$0 - f(x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_2)$$

$$\Rightarrow -f(x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}x - \frac{f(x_2) - f(x_1)}{x_2 - x_1}x_2$$

$$\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1}x = \frac{f(x_2) - f(x_1)}{x_2 - x_1}x_2 - f(x_2)$$

$$\Rightarrow x = x_2 - \frac{(x_2 - x_1)}{f(x_2) - f(x_1)}f(x_2)$$

Writing x_3 for x, we get the first approx. of the root as

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f(x_2) - f(x_1)}f(x_2)$$

Then general formula for successive approx. is $(n \geq 2)$ $\boxed{x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}f(x_n)}$

Note: When $f(x_{n-1}) = f(x_n)$, then this method fails. $\Rightarrow$ This method do not converse always. But when it converses, it will converse more rapidly than the Regula-falsi method.

or, this general formula can be written as

$$\boxed{x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}, \quad n \geq 2}$$

Working rule:

(i) Let $y = f(x) = 0$ be the given eqn.

(ii) Find two neighbouring points x_1 & x_2 and find $f(x_1)$ & $f(x_2)$.

(iii) Then use the general formula

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n+1})}{f(x_n) - f(x_{n-1})}$$

putting $n = 2, 3, 4, 5, \ldots$ etc to find $x_3, x_4, x_5, x_6, \ldots$ etc which are the first, 2nd, 3rd, 4th, ..., approximation of the roots.

(iv) Find as many approximations as required.