Hence Proved.

PROPERTIES OF LSE :

The LSE have certain desirable properties which are satisfied under certain basic assumptions mentioned above. The first desirable properties of LSE are BLUE (Best Linear Unbias Estimator)

Now, the import^ thing is that to prove that LSE as BLUE, we are to show the reality or validity of the three concepts.

① PROPERTY OF LINEARITY :

An estimate is said to be linear if it is a linear function of the sample observations. That is to say, if it is determined by a linear combination of the sample data. Thus, here, we are to prove that $\hat{\alpha}$ and $\hat{\beta}$ are linear functions of the observed sample value Y_t's. This means that the least square estimates depend on on the value of sample observations Y_t only, given the assumption of LSE mentioned above. That is,

here, we are to prove that

$$\hat{\alpha} = f(Y_t) \text{ and}$$

$$\hat{\beta} = f(Y_t)$$

Proof: We know that,

$$\hat{\beta} = \frac{\Sigma x_t y_t}{\Sigma x_t^2}$$

$$= \Sigma w_t y_t$$

$$= \sum_{t=1}^{n} w_t y_t$$

$$= \sum_{t=1}^{n} w_t Y_t$$

$$= w_1 Y_1 + w_2 Y_2 + \cdots + w_n Y_n$$

Since w_t's are fixed set of values by the hypothetical process of repeated sampling, hence the value of $\hat{\beta}$ depends on the values of Y_t's..

$$\therefore \hat{\beta} = f(Y_t) \longrightarrow (1)$$

Hence, it is proved that $\hat{\beta}$ is a linear funⁿ of the sample observation Y_t.

Similarly, we can also show that

$$\hat{\alpha} = f(Y_t) \longrightarrow (2)$$

N.B. To show that

$$\hat{\alpha} = f(Y_t)$$

Proof: We know that

$$\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}$$

$$= \frac{1}{n}\Sigma Y_t - \Sigma w_t Y_t \bar{X}$$

$$\Rightarrow \hat{\alpha} = \sum \left(\frac{1}{n} - w_t \bar{X}\right) Y_t$$

Since by the assumption $\bar{X}$ and w_t's are fixed constant from sample to sample, hence the value of $\hat{\alpha}$ depends only on the values of sample observations Y_t's i.e, $\hat{\alpha}$ is a linear funⁿ of Y_t.

$$\therefore \hat{\alpha} = f(Y_t)$$

Hence proved.

② PROPERTY OF UNBIASNESS :

The bias of an estimate is defined as the difference between its expected value and the true parameters. Thus, the bias of $\hat{\beta} = E(\hat{\beta}) - \beta$

An estimate is unbias if the bias is zero. i.e, if $E(\hat{\beta}) - \beta = 0$

$$\Rightarrow E(\hat{\beta}) = \beta$$

Similarly, $\hat{\alpha}$ will be said to be unbias if

$$E(\hat{\alpha}) - \alpha = 0$$

$$\Rightarrow E(\hat{\alpha}) = \alpha$$

Here, we are to show that

$$E(\hat{\beta}) = \beta \text{ and}$$

$$E(\hat{\alpha}) = \alpha$$

Proof: We have,

$$\hat{\beta} = \sum w_t Y_t$$

$$= \sum w_t (\alpha + \beta X_t + U_t)$$

$$= \alpha \sum w_t + \beta \sum w_t X_t + \sum w_t U_t$$

$$= \alpha . 0 + \beta . 1 + \sum w_t U_t \qquad \left[\because \sum w_t = 0 \text{ and } \sum w_t X_t = 1\right.$$

$$\Rightarrow \hat{\beta} = \beta + \sum w_t U_t \longrightarrow ③$$

$$\Rightarrow \hat{\alpha} = \sum \left(\frac{1}{n} - w_t \bar{X}\right) Y_t$$

Since by the assumption $\bar{X}$ and w_t's are fixed constant from sample to sample, hence the value of $\hat{\alpha}$ depends only on the values of sample observations Y_t's i.e., $\hat{\alpha}$ is a linear funcⁿ of Y_t.

$$\therefore \hat{\alpha} = f(Y_t)$$

Hence proved.

② PROPERTY OF UNBIASNESS :

The bias of an estimate is defined as the difference between its expected value and the true parameters. Thus, the bias of $\hat{\beta} = E(\hat{\beta}) - \beta$

An estimate is unbias if the bias is zero. i.e., if $E(\hat{\beta}) - \beta = 0$

$$\Rightarrow E(\hat{\beta}) = \beta$$

Similarly, $\hat{\alpha}$ will be said to be unbias if

$$E(\hat{\alpha}) - \alpha = 0$$

$$\Rightarrow E(\hat{\alpha}) = \alpha$$

Here, we are to show that

$$E(\hat{\beta}) = \beta \text{ and}$$

$$E(\hat{\alpha}) = \alpha$$

Proof: We have,

$$\hat{\beta} = \sum w_t Y_t$$

$$= \sum w_t (\alpha + \beta X_t + U_t)$$

$$= \alpha \sum w_t + \beta \sum w_t X_t + \sum w_t U_t$$

$$= \alpha \cdot 0 + \beta \cdot 1 + \sum w_t U_t \qquad \left[\because \sum w_t = 0 \text{ and } \sum w_t X_t = 1\right.$$

$$\Rightarrow \hat{\beta} = \beta + \sum w_t U_t \longrightarrow ③$$

Now, taking expectation on both sides, we get,

$$E(\hat{\beta}) = E(\hat{\beta}) + E(\Sigma w_t u_t)$$

$$= \beta + \Sigma w_t \, E(u_t)$$

$$= \beta \qquad (\because E(u_t) = 0 \;\forall\, t)$$

$$\therefore E(\hat{\beta}) = \beta \longrightarrow (4)$$

Hence, it is proved that $\hat{\beta}$ is an unbias estimate of β.

Similarly, we can also prove that $\hat{\alpha}$ is an unbias estimate of $\hat{\alpha}$. i.e.,

$$E(\hat{\alpha}) = \alpha \longrightarrow (5)$$

(4) PROPERTY OF MINIMUM VARIANCE :

An estimate is the best if it has the smallest variance as compared to any other estimate obtained from any other (esti- -mate obtained from) econometric method. i.e.,

$$Var(\hat{\beta}) < Var(\hat{\hat{\beta}})$$

where, $\hat{\hat{\beta}}$ is the estimate of β obtained by other econometric method (Not by OLS)

Now,

$$Var(\hat{\beta}) = E\left[\hat{\beta} - E(\hat{\beta})\right]^2$$

$$= E\left[\hat{\beta} - \beta\right]^2 \qquad (\because E(\hat{\beta}) = \beta)$$

$$= E\left(\Sigma w_t u_t\right)^2, \quad [\text{From eq}^n \; (3)]$$

$$= E\left(\sum_{t=1}^{n} w_t^2 u_t^2 + 2\sum_{t \neq t'} w_t u_t \, w_{t'} u_{t'}\right)$$

$$= \sum_{t=1}^{n} w_t^2 E(u_t^2) + 2\sum_{t\neq t'} w_t w_{t'} E(u_t u_{t'})$$

$$= \Sigma w_t^2 \sigma_u^2 + 0 \qquad \left[\because \; E(u_t^2) = \sigma_u^2, \; E(u_t u_{t'}) = 0, \;\forall\, t \neq t'\right]$$

$\Rightarrow Var(\hat{\beta}) = \sigma_u^2 \sum w_i^2$

$\Rightarrow Var(\hat{\beta}) = \sigma_u^2 \frac{1}{\sum x_i^2} \longrightarrow (6) \quad \left(\because \sum w_i^2 = \frac{1}{\sum x_i^2}\right)$

– is the OLS variance of $\hat{\beta}$ i.e. the OLS variance of the estimate of β.

$$\left(\sum w_t u_t\right)^2$$

$$= (w_1u_1 + w_2u_2 + w_3u_3 + \cdots w_nu_n)(w_1u_1 + w_2u_2 + \cdots w_nu_n)$$

$$= w_1^2u_1^2 + w_1u_1w_2u_2 + w_1u_1w_3u_3 + \cdots w_1u_1w_nu_n + w_2u_2 w_1u_1 + w_2^2u_2^2 + \cdots w_2u_2w_nu_n + w_nu_nw_1u_1 + w_nu_nw_2u_2 + \cdots w_n^2u_n^2$$

$$= w_1^2u_1^2 + w_2^2u_2^2 + \cdots w_n^2u_n^2 + 2w_1u_1w_2u_2 + \cdots 2w_3u_3w_nu_n$$

$$= \sum_{t=1}^{n} w_t^2 u_t^2 + 2\sum_{t \neq t'} w_t u_t w_{t'} u_{t'}$$

Now, to find whether $Var(\hat{\beta})$ is minimum or not, we are to estimate β by other method (not by OLS)

Let $\hat{\hat{\beta}}$ be the estimate of β obtained by other econometric method. Again, let us assume that the arbitrary estimate of β obtained by other econometric method be such that

$$\hat{\hat{\beta}} = \sum c_t Y_t \longrightarrow (7)$$

where, $c_t = w_t + d_t$, w_t being the weights and d_t is an arbitrary set of weights.

Now, $\hat{\hat{\beta}} = \sum c_t(\alpha + \beta X_t + u_t)$

$$\Rightarrow \hat{\hat{\beta}} = \alpha \sum c_t + \beta \sum c_t X_t + \sum c_t u_t \longrightarrow (8)$$

The new estimate $\hat{\hat{\beta}}$ is also assume to be an unbias estimate of β i.e., $E(\hat{\hat{\beta}}) = \beta$.

Now, taking expectation on both sides of (8), we get,

$$E(\hat{\beta}) = E\left(\alpha \sum c_t + \beta \sum c_t X_t + \sum c_t u_t\right) \longrightarrow (9)$$

Now, in (9), $E(\hat{\beta}) = \beta$ iff $\sum c_t = 0$ and $\sum c_t X_t = 1$.

Since, $\sum c_t = 0 \Rightarrow \sum (w_t + d_t) = 0$

$$\Rightarrow \sum w_t + \sum d_t = 0$$

$$\Rightarrow \sum w_t = 0 \text{ and } \sum d_t = 0$$

Thus, we have,

$$\left.\begin{array}{lcl} \sum c_t = 0 & & \sum c_t X_t = 1 \\ \sum d_t = 0 & \text{and} & \sum w_t d_t = 0 \\ \sum w_t = 0 & & \end{array}\right\} \rightarrow (10)$$

Therefore, from (8), we can have,

$$\hat{\beta} = \alpha.0 + \beta.1 + \sum c_t u_t$$

$$\Rightarrow \hat{\beta} = \beta + \sum c_t u_t \longrightarrow (11)$$

Now,

$$Var(\hat{\beta}) = E\left[\hat{\beta} - E(\hat{\beta})\right]^2$$

$$= E\left[\hat{\beta} - \beta\right]^2 \quad , \left(\because E(\hat{\beta}) = \beta\right)$$

$$= E\left(\sum c_t u_t\right)^2 \quad , \left[\text{From eq}^n \ (11)\right]$$

$$= E\left(\sum_{t=1}^{n} c_t^2 u_t^2 + 2 \sum_{t \neq t'} c_t u_t c_{t'} u_{t'}\right)$$

$$= \sum c_t^2 E(u_t^2) + 2 \sum_{t \neq t} c_t c_{t'} E(u_t u_{t'})$$

$$= \sum c_t^2 . \sigma_u^2 + 0 \qquad \left[\begin{array}{l}\because E(u_t^2) = \sigma_u^2, \forall t \text{ and} \\ E(u_t u_{t'}) = 0 \ \forall \ t \neq t'\end{array}\right]$$

$$= \sigma_u^2 \sum c_t^2$$

$$= \sigma_u^2 \sum (w_t + d_t)^2$$

$$= \sigma_u^2 \left(\sum w_t^2 + 2 \sum w_t d_t + \sum d_t^2\right)$$

$\Rightarrow Var(\hat{\hat{\beta}}) = \sigma_u^2 (\Sigma w_i^2 + \Sigma d_i^2) \quad (\because \Sigma w_i d_i = 0)$

$\Rightarrow Var(\hat{\hat{\beta}}) = \sigma_u^2 \Sigma w_i^2 + \sigma_u^2 \Sigma d_i^2$

$\Rightarrow Var(\hat{\hat{\beta}}) = \sigma_u^2 \frac{1}{\Sigma x_i^2} + \sigma_u^2 \Sigma d_i^2$

$\Rightarrow Var(\hat{\hat{\beta}}) = Var(\hat{\beta}) + \sigma_u^2 \Sigma d_i^2 \longrightarrow (12), \left(\because Var\, \hat{\beta} = \sigma_u^2 \frac{1}{\Sigma x_i^2}\right)$

$\therefore \quad Var(\hat{\beta}) < Var(\hat{\hat{\beta}}) \longrightarrow (13)$

Since $\sigma_u^2 \Sigma d_i^2 > 0$, whatever the value of d_i may have - positive or negative.

Thus LSE of β i.e., $\hat{\beta}$ has minimum variance as compared to any other abitrary estimate of β, namely $\hat{\hat{\beta}}$. That is to say, $Var(\hat{\beta})$ is minimum.

Similarly, it can also be shown that

$Var(\hat{\alpha}) < Var(\hat{\hat{\alpha}}) \longrightarrow (14)$

where, $\hat{\hat{\alpha}}$ is an arbitrary estimate of α obtained by other econometric method.

Hence, it is proved that the LSEs are best.

Therefore, from all the above three properties it is proved that the LSEs are BLUE.