LEAST SQUARE ESTIMATES OF THE TWO-VARIABLE LRM:

In estimating the parameters of LRM, generally we use the ordinary least square principle to get the best estimate of the parameters involved in the model. The least square principle implies the minimisation of the residual sum of squares i.e., the sum of the squares of the residuals (RSS) — which is the difference between actual regression line and its estimated regression line.

Let $\hat{\alpha}$ and $\hat{\beta}$ be the two arbitrary estimates of the two unknown parameters α and β respectively. Therefore, the estimated regression line will be

$$\hat{Y}_t = \hat{\alpha} + \hat{\beta} X_t \longrightarrow (3)$$

[The eq^n of the best fitted line.]

Now,

$$RSS = \sum_{t=1}^{n} u_t^2 = \sum_{t=1}^{n} (Y_t - \hat{Y}_t)^2 = \sum_{t=1}^{n} (Y_t - \hat{\alpha} - \hat{\beta} X_t)^2 \longrightarrow (4)$$

We know that for minimisation the first order necessary condition requires that —

$$\frac{\partial \sum u_t^2}{\partial \hat{\alpha}} = 0 \longrightarrow (A)$$

$$\frac{\partial \sum u_t^2}{\partial \hat{\beta}} = 0 \longrightarrow (B)$$

Now, from (A),

$$\frac{\partial \sum u_t^2}{\partial \hat{\alpha}} = \frac{\partial \sum (Y_t - \hat{\alpha} - \hat{\beta} X_t)^2}{\partial \hat{\alpha}} = 0$$

$$\Rightarrow \frac{\partial \sum (Y_t - \hat{\alpha} - \hat{\beta} X_t)^2}{\partial \hat{\alpha}} = 0$$

$$\Rightarrow 2\Sigma(Y_t - \hat{\alpha} - \hat{\beta}X_t)(-1) = 0$$

$$\Rightarrow -2\Sigma(Y_t - \hat{\alpha} - \hat{\beta}X_t) = 0$$

$$\Rightarrow \Sigma(Y_t - \hat{\alpha} - \hat{\beta}X_t) = 0 \quad \text{(dividing both sides by -2)}$$

$$\Rightarrow \Sigma Y_t - n\hat{\alpha} - \hat{\beta}\Sigma X_t = 0$$

$$\Rightarrow \Sigma Y_t = n\hat{\alpha} + \hat{\beta}\Sigma X_t$$

$$\Rightarrow \frac{1}{n}\Sigma Y_t = \hat{\alpha} + \hat{\beta}\cdot\frac{1}{n}\Sigma X_t, \quad \text{[dividing both sides by 'n']}$$

$$\Rightarrow \overline{Y} = \hat{\alpha} + \hat{\beta}\overline{X}$$

$$\Rightarrow \hat{\alpha} = \overline{Y} - \hat{\beta}\overline{X} \longrightarrow (5)$$

— This is the estimate of α.

Again from (B),

$$\frac{\partial \Sigma u_t^2}{\partial \hat{\beta}} = \frac{\partial \Sigma(Y_t - \hat{\alpha} - \hat{\beta}X_t)^2}{\partial \hat{\beta}} = 0$$

$$\Rightarrow \frac{\partial \Sigma(Y_t - \hat{\alpha} - \hat{\beta}X_t)^2}{\partial \hat{\beta}} = 0$$

$$\Rightarrow 2\Sigma(Y_t - \hat{\alpha} - \hat{\beta}X_t)(-X_t) = 0$$

$$\Rightarrow \Sigma(Y_t - \hat{\alpha} - \hat{\beta}X_t)(X_t) = 0, \quad \text{(dividing both sides by -2)}$$

$$\Rightarrow \Sigma X_tY_t - \hat{\alpha}\Sigma X_t - \hat{\beta}\Sigma X_t^2 = 0$$

$$\Rightarrow \Sigma X_tY_t - (\overline{Y} - \hat{\beta}\overline{X})\Sigma X_t - \hat{\beta}\Sigma X_t^2 = 0$$

$$\Rightarrow \Sigma X_tY_t - \overline{Y}\Sigma X_t + \hat{\beta}\overline{X}\Sigma X_t - \hat{\beta}\Sigma X_t^2 = 0$$

$$\Rightarrow \hat{\beta}\overline{X}\Sigma X_t - \hat{\beta}\Sigma X_t^2 = \overline{Y}\Sigma X_t - \Sigma X_tY_t$$

$$\Rightarrow \hat{\beta}(\overline{X}\Sigma X_t - \Sigma X_t^2) = \overline{Y}\Sigma X_t - \Sigma X_tY_t$$

$$\Rightarrow \hat{\beta} = \frac{\overline{Y}\Sigma X_t - \Sigma X_tY_t}{\overline{X}\Sigma X_t - \Sigma X_t^2}$$

$$= \frac{\Sigma X_tY_t - \overline{Y}\Sigma X_t}{\Sigma X_t^2 - \overline{X}\Sigma X_t}$$

$$= \frac{\Sigma X_t Y_t - \bar{Y}\Sigma X_t - n\bar{X}\bar{Y} + n\bar{X}\bar{Y}}{\Sigma X_t^2 - 2\bar{X}\Sigma X_t + \bar{X}\Sigma X_t}$$

$$= \frac{\Sigma X_t Y_t - \bar{Y}\Sigma X_t - \bar{X}\Sigma Y_t + \Sigma \bar{X}\bar{Y}}{\Sigma X_t^2 - 2\bar{X}\Sigma X_t + \Sigma \bar{X}^2}$$

$$= \frac{\Sigma (X_t Y_t - \bar{Y} X_t - \bar{X} Y_t + \bar{X}\bar{Y})}{\Sigma (X_t^2 - 2\bar{X} X_t + \bar{X}^2)}$$

$$= \frac{\Sigma \{ X_t (Y_t - \bar{Y}) - \bar{X}(Y_t - \bar{Y}) \}}{\Sigma (X_t - \bar{X})^2}$$

$$= \frac{\Sigma (Y_t - \bar{Y})(X_t - \bar{X})}{\Sigma (X_t - \bar{X})^2}$$

$$\therefore \hat{\beta} = \frac{\Sigma (X_t - \bar{X})(Y_t - \bar{Y})}{\Sigma (X_t - \bar{X})^2} \longrightarrow (6)$$

$n\bar{X}\bar{Y} = \bar{X} n\bar{Y}$
$= \bar{X}\Sigma Y_t$
$\bar{Y} = \frac{1}{n}\Sigma Y_t$
$\Rightarrow n\bar{Y} = \Sigma Y_t$
$n\bar{X}\bar{Y} = \Sigma \bar{X}\bar{Y}$
$\bar{X} = \frac{1}{n}\Sigma X_t$
$\Rightarrow \Sigma X_t = n\bar{X}$
$\bar{X}\Sigma X_t$
$= \bar{X} n\bar{X}$
$= n\bar{X}^2$
$= \Sigma \bar{X}^2$

Now, if we consider $(X_t - \bar{X}) = x_t$ and $(Y_t - \bar{Y}) = y_t$, then from (6), we get,

$$\hat{\beta} = \frac{\Sigma x_t y_t}{\Sigma x_t^2} \longrightarrow (7)$$

— this is the OLS estimates of β.

Now, if we take $\frac{x_t}{\Sigma x_t^2} = \Sigma w_t$, then from (7), we get,

$$\hat{\beta} = \Sigma w_t y_t \longrightarrow (8)$$

Now, from (8), we can get,

$$\hat{\beta} = \Sigma w_t (Y_t - \bar{Y}), \quad (\because y_t = Y_t - \bar{Y})$$

$$= \Sigma w_t Y_t - \bar{Y}\Sigma w_t$$

$$= \Sigma w_t Y_t - \bar{Y} \cdot \Sigma \left(\frac{x_t}{\Sigma x_t^2}\right)$$

$$= \Sigma w_t Y_t - \bar{Y}\frac{\Sigma x_t}{\Sigma x_t^2}$$

$$= \Sigma w_t Y_t - \bar{Y}\,\frac{\Sigma(X_t-\bar{X})}{\Sigma x_t^2} \qquad (\because x_t = X_t - \bar{X})$$

$$= \Sigma w_t Y_t - \bar{Y}\,\frac{0}{\Sigma x_t^2}$$

$\Big[\because \Sigma(X_t-\bar{X}) = 0$ as the sum of the deviations of the observation from their A.M. is always equal to 0

$$\Rightarrow \hat{\beta} = \Sigma w_t Y_t \longrightarrow (9)$$

Thus, we have,

$$\hat{\beta} = \frac{\Sigma(X_t-\bar{X})(Y_t-\bar{Y})}{\Sigma(X_t-\bar{X})^2} \longrightarrow (6)$$

$$= \frac{\Sigma x_t y_t}{\Sigma x_t^2} \longrightarrow (7)$$

$$= \Sigma w_t y_t \longrightarrow (8)$$

$$= \Sigma w_t Y_t \longrightarrow (9)$$

OLS Estimates of β.

Ques: Take a two-variable LRM of the form

$$Y_t = \alpha + \beta X_t + u_t$$

Specify the model. Mention the basic assumption of LRM or LSE. Also obtain the OLS estimate of the parameter α and β.