Numerical Integration:

Numerical integration is the process of computing the value of a definite integral from the tabulated values of the integrand. If it is applied to the integration of a function of a single variable, the process is known as quadrature. The expression for the integral so obtained is called a quadrature formula

The accuracy of the quadrature formula depends on the following factors:

(i) the size of the interval,

(ii) the range of the integration,

(iii) the degree of the polynomial,

and (iv) the points through which it passes.

Numerical integration :-

General Quadrature formula :-

[OR ; Newton-Cote's Quadrature formula]

Let $y = f(x)$ be a function.

To find $\int_a^b f(x)\,dx$ or $\int_a^b y\,dx$

Let us divide $[a, b]$ into n equal parts by the points $a = x_0$, $a+h = x_1$, $a+2h = x_2$, ..., $x_n = a + nh = b$; each of length h.

Let $y_0, y_1, y_2, \ldots\ldots, y_n$ be the values of the function corresponding the values $x_0, x_1, \ldots x_n$ of x.

$$\therefore \int_a^b y\,dx = \int_{x_0}^{x_n} y\,dx.$$

Let $u = \frac{x - x_0}{h} \Rightarrow uh = x - x_0 \Rightarrow h\,du = dx$

& $x = x_0 + uh$

Now,

$$\int_a^b y\,dx = \int_{x_0}^{x_n} y_x\,dx$$

$$= \int_0^n y_{x_0+uh}\,h\,du$$

$$= h\int_0^n y_{x_0+uh}\,du$$

$u = \frac{x_n - x_0}{h} = \frac{x_0 + nh - x_0}{h} = n$

$$= h\int_0^n \Big[y_0 + u\,\Delta y_0 + \frac{u(u-1)}{\lfloor 2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{\lfloor 3}\Delta^3 y_0 + \cdots \text{ upto } n \text{ terms} \Big]\,du$$

(by Newton's forward int. formula)

$$= h\int_0^n \Big[y_0 + u\,\Delta y_0 + \frac{u^2 - u}{2}\Delta^2 y_0 + \frac{u^3 - 3u^2 + 2u}{6}\Delta^3 y_0 + \cdots \Big]du$$

$$\int_a^b y\,dx = h\left[u y_0 + \frac{u^2}{2}\Delta y_0 + \frac{\frac{u^3}{3} - \frac{u^2}{2}}{2}\Delta^2 y_0 + \frac{\frac{u^4}{4} + \frac{3u^3}{3} + 2\cdot\frac{u^2}{2}}{6}\cdot\Delta^3 y_0 \cdots \right]_0^n$$

$$= h\left\{ n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{\frac{n^3}{3} - \frac{n^2}{2}}{2}\Delta^2 y_0 + \frac{\frac{n^4}{4} - n^3 - n^2}{6}\cdot\Delta^3 y_0 + \cdots \right]$$

which is General quadrature formula.

Trapezoidal rule:

Putting $n=1$ in general quadrature formula and neglecting second and higher differences, we get

$$\int_{x_0}^{x_0+h} y\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0 \right], \quad \because \; x_n = x_0 + nh = x_0 + h,\ n=1$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0) \right]$$

$$= \frac{h}{2}\left[2y_0 + y_1 - y_0 \right]$$

$$= \frac{h}{2}(y_0 + y_1)$$

Slly,

$$\int_{x_0+h}^{x_0+2h} y\,dx = \frac{h}{2}\left[y_1 + y_2 \right]$$

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$$\int_{x_0+(n-1)h}^{x_0+nh = x_n} y\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

$$\therefore \int_{x_0}^{x_0+nh} y\,dx + \int_{x_0+h}^{x_0+2h} y\,dx + \cdots + \int_{x_0+(n-1)h}^{x_0+nh} y\,dx$$

$$= \frac{h}{2}\left[(y_0 + y_1) + (y_1 + y_2) + \cdots + (y_{n-1} + y_n) \right]$$

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$$\Rightarrow \int_{x_0}^{x_0+nh} y\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right]$$

Note: In the trapezoidal rule we take y as a function of x of degree '1'. i.e. of the form $y = ax + b$.
