12/9/92

Numerical Differentiation.

It is the process of calculating the derivative of a function by means of a set of given values of that function. The problem is solved by representing the function by an interpolation formula and then differentiating this formula as many times as we desire.

Ex. Given the following pairs of values of x and $y = f(x)$.

x	1	2	4	8	10
y	0	1	5	21	27

Determine numerically the 1st and 2nd derivative of $f(x)$ at $x = 4$. [Ans: $\left.\frac{dy}{dx}\right]_{x=4} = 2.833$ (app.), $\frac{d^2y}{dx^2} = \left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right]_{x=4} = 0.861$

Solⁿ. [Here arguments are not equally distributed. So we apply Newton's d.d. formula.]

First we construct the divided difference table.

[Here $x = 4$ is given in the table. So we write $x = 4$ at last]

x	y				
$x_0 = 1$	$y_0 = 0$				
		$\delta(x_0, x_1) = \frac{y_1 - y_0}{x_1 - x_0} = 1$			
$x_1 = 2$	$y_1 = 1$		$\delta(x_0, x_1, x_2) = \frac{1}{3}$		
		$\delta(x_1, x_2) = \frac{10}{3}$		$\delta(x_0, x_1, x_2, x_3) = -\frac{1}{24}$	
$x_2 = 8$	$y_2 = 21$		$\delta(x_1, x_2, x_3) = -\frac{1}{24}$		$= \frac{-1}{144}$
		$\delta(x_2, x_3) = 3$		$\delta(x_1, x_2, x_3, x_4) = -\frac{1}{16}$	
$x_3 = 10$	$y_3 = 27$		$\delta(x_2, x_3, x_4) = -\frac{1}{16}$		
		$\delta(x_3, x_4) = \frac{11}{3}$			
$x_4 = 4$	$y_4 = 5$				

Newton's d.d formula is

$$y = y_0 + (x-x_0)\delta(x_0,x_1) + (x-x_0)(x-x_1)\,\delta(x_0,x_1,x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2)\,\delta(x_0,x_1,x_2,x_3)$$
$$+ (x-x_0)(x-x_1)(x-x_2)(x-x_3)\,\delta(x_0,x_1,x_2,x_3,x_4)$$

$$\therefore \frac{dy}{dx} = \delta(x_0,x_1) + \left[(x-x_0)+(x-x_1)\right]\delta(x_0,x_1,x_2)$$
$$+\left[(x-x_0)(x-x_1) + (x-x_0)(x-x_2) + (x-x_1)(x-x_2)\right]\delta(x_0,x_1,x_2,x_3)$$
$$+\left[(x-x_0)(x-x_1)(x-x_2) + (x-x_0)(x-x_1)(x-x_3)\right.$$
$$\left. + (x-x_0)(x-x_2)(x-x_3) + (x-x_1)(x-x_2)(x-x_3)\right]\delta(x_0,x_1,x_2,x_3,x_4)$$

$$\left.\frac{dy}{dx}\right] = 1 + \left[(x-1)+(x-2)\right]\frac{1}{3} + \left[(x-1)(x-2) + (x-1)(x-8) + (x-2)(x-8)\right]\left(-\frac{1}{24}\right)$$
$$+\left[(x-1)(x-2)(x-8) + (x-1)(x-2)(x-10) + (x-1)(x-8)(x-10)\right.$$
$$\left. + (x-2)(x-8)(x-10)\right]\left(-\frac{1}{144}\right)$$

$$\left.\frac{dy}{dx}\right]_{x=4} = 1 + \left[3+2\right]\left(\frac{1}{3}\right) + \left\{3\cdot 2 + 3\cdot(-4) + 2(-4)\right\}\left(-\frac{1}{24}\right)$$
$$+\left[3\cdot 2(-4) + 3\cdot 2(-6) + 3\cdot(-4)(-6) + (2)(-4)-6\right]\left(-\frac{1}{144}\right)$$
$$= 1 + 1.66 + .583 + .416 = 2.827$$ (correct.)

$\frac{d^2}{}=$

Ex. Find $\frac{dy}{dx}$ at $x=0.5$ from the following table.

x	0	1	2	3	4
y	2	3	6	11	18

[Here arguments are equally distributed] table. So use

Also, $x=0.5$ is beginning of the

Newton's forward Int. formula]

Sol^n. First we construct the difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	2				
		1			
1	3		2		
		3		0	
2	6		2		0
		5		0	
3	11		2		
		7			
4	18				

We write Newton's forward interpolation formula,

$$y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 \quad , \text{ where, } u = \frac{x-x_0}{h} = \cdot 5$$

$(h=1)$

$\therefore \frac{du}{dx} = \frac{1}{h}$

Now, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$= \frac{1}{h}\frac{dy}{du} = \frac{1}{h}\frac{d}{du}\left[y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0\right]$$

$$= \frac{1}{h}\left[1 \cdot \Delta y_0 + u\,\Delta^2 y_0 - \frac{1}{2}\Delta^2 y_0\right]$$

$$= \frac{1}{1}\left[1 \cdot 1 + \cdot 5 \cdot 2 - \frac{1}{2} \cdot 2\right]$$

$$= 1. \quad \leftarrow \text{Ans}$$