§. Central Difference Interpolation formulae:

(For interpolation near the middle of a difference table, central difference formulae are preferred.)

(1) Gauss's forward interpolation formulae:

Newton's general interpolation formula for divided difference is

$$y = y_0 + (x-x_0)\delta(x_0, x_1) + (x-x_0)(x-x_1)\delta(x_0, x_1, x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2)\delta(x_0, x_1, x_2, x_3)$$
$$+ \cdots\cdots\cdots$$
$$+ (x-x_0)(x-x_1)\cdots(x-x_{n-1})\delta(x_0, x_1, \cdots x_n)$$
$$+ (x-x_0)(x-x_1)\cdots(x-x_n)\delta(x, x_0, x_1, \cdots x_n). \longrightarrow (1)$$

In (1), we substitute

$x_0 = x_0$, $x_1 = x_0 + h$, $x_2 = x_0 - h$,
$x_3 = x_0 + 2h$, $x_4 = x_0 - 2h$,
$x_5 = x_0 + 3h$, $x_6 = x_0 - 3h$, etc.

$x_4 = x_0 - 2h$	y
$x_2 = x_0 - h$	y
x_0	y_0
$x_1 = x_0 + h$	y_1
$x_3 = x_0 + 2h$	y_2
$x_5 = x_0 + 3h$	y_3
$x_7 = x_0 + 4h$	y_4

etc

Then we get,

$$y = y_0 + (x-x_0)\delta(x_0, x_0+h)$$
$$+ (x-x_0)(x-x_0-h)\delta(x_0, x_0+h, x_0-h)$$
$$+ (x-x_0)(x-x_0-h)(x-x_0+h)\delta(x_0, x_0+h, x_0-h, x_0+2h)$$
$$+ (x-x_0)(x-x_0-h)(x-x_0+h)(x-x_0-2h)\delta(x_0, x_0+h, x_0-h, x_0+2h, x_0-2h)$$
$$+ \cdots\cdots\cdots \longrightarrow (2)$$

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We put $u = \dfrac{x - x_0}{h}$ or $x - x_0 = hu$.

Then from **(2)**, we have

$$y = y_0 + hu\,\delta(x_0, x_0+h) + hu.(hu-h)\,\delta(x_0, x_0+h, x_0-h)$$
$$+ hu(hu-h)(hu+h)\,\delta(x_0, x_0+h, x_0-h, x_0+2h)$$
$$+ hu(hu-h)(hu+h)(hu-2h)\,\delta(x_0, x_0+h, x_0-h, x_0+2h, x_0-2h)$$
$$+ \ldots\ldots\ldots$$

or,
$$y = y_0 + hu\,\delta(x_0, x_0+h) + h^2u(u-1)\,\delta(x_0-h, x_0, x_0+h)$$
$$+ h^3u(u^2-1)\,\delta(x_0-h, x_0, x_0+h, x_0+2h)$$
$$+ h^4u(u^2-1)(u-2)\,\delta(x_0-2h, x_0-h, x_0, x_0+h, x_0+2h)$$
$$+ \ldots\ldots\ldots \longrightarrow (3)$$

But we have the relation between divided difference and simple difference as

$\left[f(x_0, x_1, \ldots, x_n) = \dfrac{\Delta^n}{n!\,h^n} f(x_0)\right]$

$$\delta^n y_k = \delta(x_k + nh,\ x_k + (n-1)h,\ \ldots\ldots,\ x_k + h,\ x_k)$$

(just a symbol) number of points (n+1) n th diff

$$= \frac{\Delta^n y_k}{n!\,h^n},\quad k = 0, 1, 2, \ldots\ldots$$

no. of pt (n+1) ⇒ n-th order difference

Thus, we have,

[no. of pts. two → first diff; three → 2nd "; four → 3rd "]

$$\delta(x_0, x_0+h) = \frac{\Delta^1 y_0}{h}$$
$$\delta(x_0-h, x_0, x_0+h) = \frac{\Delta^2 y_{-1}}{2!\,h^2}$$
$$\delta(x_0-h, x_0, x_0+h, x_0+2h) = \frac{\Delta^3 y_{-1}}{3!\,h^3}$$
$$\delta(x_0-2h, x_0-h, x_0, x_0+h, x_0+2h) = \frac{\Delta^4 y_{-2}}{4!\,h^4}$$
$$\delta(x_0-2h, x_0-h, x_0, x_0+h, x_0+2h, x_0+3h) = \frac{\Delta^5 y_{-2}}{5!\,h^5}$$
$$\ldots\ldots$$

x_0-3h	y_{-3}
x_0-2h	y_{-2}
x_0-h	y_{-1}
x_0	y_0
x_0+h	y_1
x_0+2h	y_2
x_0+3h	y_3

Substituting in (3), we get,

$$y = y_0 + hu \cdot \frac{\Delta' y_0}{h} + h^2 u(u-1) \cdot \frac{\Delta^2 y_{-1}}{2!\,h^2}$$

$$+ h^3 u(u^2-1) \frac{\Delta^3 y_{-1}}{3!\,h^3} + h^4 u(u^2-1)(u-2) \cdot \frac{\Delta^4 y_{-2}}{4!\,h^4}$$

$$+ h^5 u(u^2-1)(u^2-2^2) \frac{\Delta^5 y_{-2}}{5!\,h^5} + \cdots$$

or, $$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{u(u^2-1)(u-2)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{u(u^2-1)(u^2-2^2)}{5!}\Delta^5 y_{-2} + \cdots$$

$$+ \cdots$$

Which is Gauss's forward interpolation formula.