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Laplace Transform

INTRODUCTION

Pierre-Simon Laplace was the first one to use the Laplace transform in his work of probability and wrote extensively about the use of generating functions in 'Essaiphilosophique sur les probabilités' published in 1814, and the integral form of the Laplace transform evolved naturally as a result. The theory was further developed in the 19th and early 20th centuries by Mathias Lerch, Oliver Heaviside, and Thomas Bromwich. It is a powerful technique to solve ordinary and partial differential equations, which transforms the differential equation into an elementary algebraic expression that can be simply transformed back into the solution of the original problem.

1. LAPLACE TRANSFORM

The Laplace transform of a function $f(t)$ for $t \geq 0$ is denoted by $L[f(t)]$ or by $F(s)$, and is defined by the following equation

$$L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt \qquad \text{...(1.1)}$$

The above integral defining the Laplace transform is an improper integral that may converge or diverge, depending on the integrand. If the integral defined by equation (1.1) is convergent then it is said that the function $f(t)$ possesses a Laplace transform, otherwise the transform diverges. The parameter s belongs to some domain (the set of all possible inputs for the function.) on the real line or in the complex plane and is chosen appropriately so as to ensure the convergence of the Laplace integral (1.1). In a mathematical and technical sense, the domain of s is quite important. However, in a practical sense, when differential equations are solved, the domain of s is mostly ignored.

1.1. CONDITIONS FOR LAPLACE TRANSFORM TO EXIST

The Laplace transform of a function $f(t)$ exists only if the function is

(*i*) An arbitrary piecewise continuous function in every finite interval and $f(t) = 0$ for all negative values of t

(*ii*) of exponential order.

Here the function $f(t)$ is said to be piecewise continuous function on the interval $[0, \infty)$ if

a. $\lim_{t \to 0^+} f(t) = f(0^+)$ exists and

b. $f(t)$ is continuous on every finite interval $(0, b)$ except at a finite number of discontinuities.

Proof : Let the Laplace transform of $f(t)$ for $t_0 > 0$ can be expressed as

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{t_0} f(t)e^{-st}dt + \int_{t_0}^{\infty} f(t)e^{-st}dt$$

Continuity of $f(t)$ in the finite interval $(0, t_0)$ implies that $\int_0^{t_0} f(t)e^{-st}dt$ exists. Further if $f(t)$ is of exponential order, then $\lim_{t\to\infty} f(t)\, e^{-s_0 t}$ is finite.

But for a function to be finite, the function should never approach infinite and should always be smaller than some value M a real constant, such that $|f(t)e^{-s_0 t}| < M$ for all values of $t \geq t_0$.

$\Rightarrow \ |f(t)| < Me^{s_0 t}$ for all values of $t \geq t_0$

Considering the Laplace transform of $f(t)$, one gets

$$\left|\int_{t_0}^{\infty} f(t)e^{-st}dt\right| \leq \int_{t_0}^{\infty} |f(t)|\, e^{-st}dt$$

Using the condition $|f(t)| < Me^{s_0 t}$, the above Laplace transform could be rewritten as

$$\left|\int_{t_0}^{\infty} f(t)e^{-st}dt\right| < \int_{t_0}^{\infty} Me^{s_0 t}e^{-st}dt = M\int_{t_0}^{\infty} e^{-(s-s_0)t}dt$$

$$= -M\left.\frac{e^{-(s-s_0)t}}{s-s_0}\right|_{t_0}^{\infty} = -M\left(0 - \frac{e^{-(s-s_0)t_0}}{s-s_0}\right) = M\frac{e^{-(s-s_0)t_0}}{s-s_0} \text{ if } s > s_0$$

$$\left|\int_{t_0}^{\infty} f(t)\, e^{-st}dt\right| < M\frac{e^{-(s-s_0)t_0}}{s-s_0} \text{ if } s > s_0$$

Thus the resultant $M\frac{e^{-(s-s_0)t}}{s-s_0}$ is finite leading to the conclusion that the laplace transform exists on the interval $[0, \infty)$ if $s > s_0$ and $f(t) \to Me^{s_0 t}$. To understand the concept consider the laplace transform of $f(t) = t^2$, such that $t^2 \to Me^{s_0 t}$ at $t \geq t_0$ so that the function behaves as an exponential function for all values of $t \geq t_0$, thus the Laplace transform of the function exists. However it could not be said for the function $f(t) = e^{t^3}$ as $e^{t^3} \nrightarrow e^t$ as $t \to \infty$, hence e^{t^3} cannot be transformed using Laplace transforms.

Example 1.1. Find the Laplace transform of the function $f(t) = 1$.

Solution. The Laplace transform of $f(t) = 1$ is given as

$$F(s) = \int_0^{\infty} 1.e^{-st}\, dt = \int_0^{\infty} e^{-st}dt = -\left.\frac{e^{-st}}{s}\right|_0^{\infty} = -\frac{e^{-\infty} - e^{-0}}{s} - \left(0 - \frac{1}{s}\right) = \frac{1}{s}$$

Note that the transform is defined only if $s > 0$, otherwise it diverges.

Example 1.2. Find the Laplace transform of the function $f(t) = t$.

Solution. The Laplace transform of $f(t) = t$ is given as

$$F(s) = \int_0^\infty t.e^{-st}\,dt = -\frac{t.e^{-st}}{s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt = 0 - \frac{e^{-st}}{s^2}\Big|_0^\infty = -\left(0 - \frac{1}{s^2}\right) = \frac{1}{s^2}$$

The transform is defined only if $s > 0$

Example 1.3. Find the Laplace transform of the function $f(t) = t^n$.

Solution. The Laplace transform of $f(t) = t^n$ is given as

$$F(s) = \int_0^\infty t^n e^{-st}dt = -\frac{t^n e^{-st}}{s}\Big|_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt = 0 + \frac{n}{s}\left(\frac{t^{n-1}.e^{-st}}{s}\Big|_0^\infty + \frac{n-1}{s}\int_0^\infty t^{n-2}e^{-st}dt\right)$$

$$= \frac{n}{s}\left(0 - \frac{n-1}{s}\left(\frac{t^{n-2}.e^{-st}}{s}\Big|_0^\infty + \frac{n-2}{s}\int_0^\infty t^{n-3}e^{-st}dt\right)\right) = \frac{n(n-1)(n-2)}{s^3}\int_0^\infty t^{n-3}e^{-st}dt$$

Proceeding in a similar way one gets

$$\int_0^\infty t^n e^{-st}dt = \frac{n!}{s^n}\int_0^\infty e^{-st}dt = -\frac{n!}{s^{n+1}}e^{-st}\Big|_0^\infty = -\frac{n!}{s^{n+1}}(0-1) = \frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}$$

The transform is defined only if $s > 0$

Example 1.4. Find the Laplace transform of the function $f(t) = e^{-t^2}$.

Solution. The Laplace transform of $f(t) = e^{-t^2}$ is given as

$$F(s) = \int_0^\infty e^{-t^2}e^{-st}dt = \int_0^\infty e^{-t^2-st}dt$$

In order to complete the square of the term $t^2 + st$, add and subtract $s^2/4$ to the exponential term so that the above integral could be rewritten as

$$F(s) = \int_0^\infty e^{-\left(t^2+st+\frac{s^2}{4}\right)}e^{\frac{s^2}{4}}dt = e^{\frac{s^2}{4}}\int_0^\infty e^{-\left(t+\frac{s}{2}\right)^2}dt = e^{\frac{s^2}{4}}\frac{\sqrt{\pi}}{2}$$

Here the integral formula $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$ has been utilized.

Example 1.5. Find the Laplace transform of the function $f(t) = e^{\alpha t}$ if it exists.

Solution. The Laplace transform of a function is given by equation $F(s) = \int_0^\infty f(t)e^{-st}dt$, using $f(t) = e^{\alpha t}$, one gets

$$F(s) = \int_0^\infty e^{\alpha t}e^{-st}dt = \int_0^\infty e^{-(s-\alpha)t}dt$$

The transform exists only if $s > \alpha$, otherwise the integral will diverge, hence

$$F(s) = \int_0^\infty e^{-(s-\alpha)t}dt = \left.\frac{e^{-(s-\alpha)t}}{s-\alpha}\right|_0^\infty = -\left(0 - \frac{1}{s-\alpha}\right) = \frac{1}{s-\alpha}$$ exists for all values of $s > \alpha$.

Example 1.6. Check if the Laplace transform of the function $f(t) = e^{t^2}$ exists.

Solution. The Laplace transform of $f(t) = e^{t^2}$ is given as

$$F(s) = \int_0^\infty e^{t^2} e^{-st} dt = \int_0^\infty e^{t^2 - st} dt$$

In order to complete the square of the term $t^2 - st$, add and subtract $s^2/4$ to the exponential term so that the above integral could be rewritten as

$$F(s) = \int_0^\infty e^{\left(t^2 - st + \frac{s^2}{4}\right)} e^{-\frac{s^2}{4}} dt = e^{-\frac{s^2}{4}} \int_0^\infty e^{\left(t-\frac{s}{2}\right)^2} dt \to \infty$$ for all values of $t - \frac{s}{2}$

Hence the Laplace transform of $f(t)$ does not exists.

Example 1.7. Find the Laplace transform of $f(t) = \cos at$.

Solution. The Laplace transform of a function is given by

$$F(s) = \int_0^\infty f(t)\, e^{-st} dt,$$ substituting $f(t) = \cos at$, one gets

$$F(s) = \int_0^\infty \cos at\, e^{-st} dt = \int_0^\infty \left(\frac{e^{iat} + e^{-iat}}{2}\right) e^{-st} dt \qquad \left(\text{using } \cos at = \frac{e^{iat} + e^{-iat}}{2}\right)$$

$$= \frac{1}{2}\int_0^\infty (e^{-(s-ia)t} + e^{-(ia+s)t})\, dt$$

$$= \frac{1}{2}\left(-\frac{e^{-(s-ia)t}}{s-ia} - \frac{e^{-(ia+s)t}}{s+ia}\right)\Bigg|_0^\infty = -\frac{1}{2}\left(0 - \frac{1}{s-ia} + 0 - \frac{1}{s+ia}\right) = \frac{1}{2}\left(\frac{1}{s-ia} + \frac{1}{s+ia}\right)$$

$$= \frac{1}{2}\frac{s + ia + s - ia}{s^2 + a^2} = \frac{s}{s^2 + a^2}$$

Alternatively

$$L(\cos at) = L\left(\frac{e^{iat} + e^{-iat}}{2}\right) = \frac{1}{2}L(e^{iat}) + \frac{1}{2}L(e^{-iat})$$ [from example 1.5]

$$= \frac{1}{2}\left[\frac{1}{s-ia} + \frac{1}{s+ia}\right]$$

$$= \frac{s}{s^2 + a^2}$$

Example 1.8. Find the Laplace transform of $f(t) = \sin at$.

Solution. The Laplace transform of a function is given by $F(s) = \int_0^\infty f(t) e^{-st} dt$

Substituting $f(t) = \sin at$, one gets

$$F(s) = \int_0^\infty \sin at\, e^{-st} dt = \int_0^\infty \left(\frac{e^{iat} - e^{-iat}}{2i}\right) e^{-st} dt \qquad \left(\text{using } \sin at = \frac{e^{iat} - e^{-iat}}{2i}\right)$$

$$= \frac{1}{2i} \int_0^\infty (e^{-(s-ia)t} - e^{-(ia+s)t})\, dt$$

$$= \frac{1}{2i}\left(-\frac{e^{-(s-ia)t}}{s-ia} + \frac{e^{-(ia+s)t}}{s+ia}\right)\Bigg|_0^\infty = -\frac{1}{2i}\left(0 - \frac{1}{s-ia} - 0 + \frac{1}{s+ia}\right) = \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right)$$

$$= \frac{1}{2i}\,\frac{s+ia-s+ia}{s^2+a^2} = \frac{a}{s^2+a^2}$$

Example 1.9. Find the Laplace transform of $f(t) = \frac{\sin t}{t}$.

Solution. The function $f(t) = \frac{\sin t}{t}$ can be written in the expanded form as

$$\frac{\sin t}{t} = \frac{1}{t} \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}$$

The Laplace transform of the function is given as

$$F(s) = \int_0^\infty \frac{\sin t}{t} e^{-st} dt = \int_0^\infty \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!} e^{-st} dt = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \int_0^\infty t^{2n} e^{-st} dt$$

Using $\int_0^\infty x^n e^{-ax} dx = \frac{\Gamma(n+1)}{a^{n+1}}$, one gets

$$= \sum_{n=0}^{\infty} (-1)^{3n} \frac{1}{(2n+1)!} \frac{\Gamma(2n+1)}{s^{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{\Gamma(2n+1)}{s^{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{(2n)!}{s^{2n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)s^{2n+1}} = \tan^{-1}\left(\frac{1}{s}\right) \qquad \left[\text{using } \tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}\right]$$

Example 1.10. Find the Laplace transform of $f(t) = \sin^2 t$.

Solution. The function $f(t) = \sin^2 t = \frac{1-\cos 2t}{2}$, so that the Laplace transform could be written as

$$L(\sin^2 t) = \int_0^\infty \frac{1-\cos 2t}{2} e^{-st}\,dt = \frac{1}{2}\left(\int_0^\infty e^{-st}\,dt - \int_0^\infty \cos 2t\, e^{-st}\,dt\right).$$

Using the result of example (1.7) and considering $a = 2$, one gets

$$L(\sin^2 t) = \frac{1}{2}\left(\frac{e^{-st}}{-s}\Bigg|_0^\infty - \frac{s}{s^2+4}\right)$$

$$= \frac{1}{2s} - \frac{s}{2(s^2+4)} = \frac{2}{s(s^2+4)}$$

Example 1.11. Find the laplace transform of $e^{-at}\sin bt$.

Solution. The Laplace transform of a function $f(t)$ is given as

$$F(s) = \int_0^\infty f(t) e^{-st}\,dt$$

Substituting $f(t) = e^{-at}\sin bt$, one gets

$$F(s) = \int_0^\infty e^{-at}\sin bt\, e^{-st}\,dt$$

Using $\sin bt = \frac{e^{ibt} - e^{-ibt}}{2i}$, the above equation can be rewritten as

$$F(s) = \frac{1}{2i}\int_0^\infty e^{-(a+s)t}\,[e^{ibt} - e^{-ibt}]\,dt$$

$$= \frac{1}{2i}\left[\int_0^\infty e^{-[(a+s)-ib]t}\,dt - \int_0^\infty e^{-[(a+s)+ib]t}\,dt\right]$$

$$= \frac{1}{2i}\left[-\frac{e^{-[(a+s)-ib]t}}{a+s-ib}\Bigg|_0^\infty + \frac{e^{-[(a+s)+ib]t}}{(a+s)+ib}\Bigg|_0^\infty\right]$$

$$= \frac{1}{2i}\left[-\left(\frac{e^{-\infty} - e^{-0}}{a+s-ib}\right) + \left(\frac{e^{-\infty} - e^{0}}{a+s+ib}\right)\right]$$

$$= \frac{1}{2i}\left[-\left(\frac{-1}{a+s-ib}\right) + \left(\frac{-1}{a+s+ib}\right)\right]$$