$\vec{p} = \iiint_V \vec{r}' \rho \, dv' = \vec{r} \iiint_V \rho \, dv' = \vec{r} q$. where

q is the charge given by $\iiint_V \rho \, dv'$.

Hence the dipole moment of a point charge q at a radial distance $\vec{r}$ is given by

$$\vec{p} = \vec{r}.q.$$

If the cartesian co-ordinates of the point are $(x\,y\,z)$ Then the dipole moment vector will have components $P_x\ P_y\ P_z$ along X, Y and Z axis respectively, given by

$$\vec{P_x} = xq\hat{i},\ \vec{P_y} = yq\hat{j} \text{ and } \vec{P_z} = zq\hat{k}$$

Poisson's and Laplace's equations from differential form of Gauss's law :—

Poisson's equation :— The differential form of Gauss's law states $\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0} \xrightarrow{\text{(in S.I unit)}} (1)$

where $\vec{E}$ is the electric field intensity and ρ the volume density of charge.

The electric field at a point is given by the negative gradient of electric potential ϕ, or

$$\vec{E} = -\text{grad}\,\phi = -\vec{\nabla}\phi$$

Substituting in (1) we have

$$\text{div}\,\vec{E} = \vec{\nabla}.\vec{E} = \vec{\nabla}.(-\vec{\nabla}\phi) = \frac{\rho}{\epsilon_0}$$

$$\text{or } -\vec{\nabla}^2\phi = \frac{\rho}{\epsilon_0}$$

$$\text{or } \vec{\nabla}^2\phi = -\frac{\rho}{\epsilon_0}$$

$$\text{or} \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) = -\frac{\rho}{\epsilon_0}.$$

This is the poisson's equation which holds good for each point in space.

Laplace equation — When the enclosed charge density is zero i.e. the point of observation is in empty space $\rho = 0$ and we get

$$\nabla^2 \phi = 0$$

$$\text{or } \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

This is Laplace's equation.

Laplace operator — The operator div grad $= \nabla^2$ is known as Laplace operator. Thus div grad $\phi = \vec{\nabla} \cdot \vec{\nabla} \phi = \nabla^2 \phi$.

Where ϕ is a scalar function, grad ϕ is a vector but div grad ϕ is a scalar. The value of $\nabla^2 \phi$ in empty space $(\rho = 0)$ is zero.

Significance — Laplace's eqn is an important differential equation and a function which satisfies this equation is known as spherical harmonic. A very important property of such function is

" The average value of function $\phi(x\,y\,z)$ satisfying Laplace equation over any spherical surface is equal to its value at the centre of the spherical surface ".

We have already seen that the electric potential and intensity at any point due to a spherical shell of charge is the same as if the whole charge were concentrated at its centre.

Q. Write down Laplace eqn and show if the functions $f(x,y) = x^2 - y^2$ and $g(x,y) = x^2+y^2$ satisfy the two dimensional Laplace equation.

Ans-> Laplace eqn states $\nabla^2\phi = 0$

Now $\nabla^2\phi = \vec{\nabla}\cdot\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

In the two dimension

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$$

(i) For $\phi = f(x,y) = x^2 - y^2$

$$\frac{\partial\phi}{\partial x} = 2x \text{ and } \frac{\partial^2\phi}{\partial x^2} = 2$$

Similarly $\frac{\partial\phi}{\partial y} = -2y$ and $\frac{\partial^2\phi}{\partial y^2} = -2$

$\therefore \nabla^2\phi = +2 - 2 = 0$

Hence $f(x,y) = x^2 - y^2$ satisfies two dimensional Laplace eqn.

(ii) For $\phi = g(x) = x^2+y^2$

$$\frac{\partial\phi}{\partial x} = 2x, \text{ and } \frac{\partial^2\phi}{\partial x^2} = 2$$

Similarly $\frac{\partial\phi}{\partial y} = 2y$ and $\frac{\partial^2\phi}{\partial y^2} = 2$

$\therefore \nabla^2\phi = 2+2 = 4 \neq 0$

$\therefore$ Function $g(x,y) = x^2+y^2$ does not satisfy the two dimensional Laplace eqn.

Q. Show that the potential function
$V = x^2 + y^2 - 2z^2$ satisfies the Laplace eqn.

Ans) Laplace eqn states $\nabla^2 V = 0$

Now $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

and $\frac{\partial V}{\partial x} = 2x$, $\frac{\partial V}{\partial y} = 2y$, $\frac{\partial V}{\partial z} = -4z$

$\Rightarrow \frac{\partial^2 V}{\partial x^2} = 2$, $\frac{\partial^2 V}{\partial y^2} = 2$, $\frac{\partial^2 V}{\partial z^2} = -4$

$\therefore \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 2 + 2 - 4 = 0$

Hence the potential function $V = x^2 + y^2 - 2z^2$ satisfy the Laplace eqn.

Q. Show that electric potential derived from electric field $x\hat{i} + y\hat{j} + 2\hat{k}$ satisfies Laplace eqn.

Ans) we have given $\vec{E} = x\hat{i} - y\hat{j} - 2\hat{k}$

Now $\vec{E} = -\vec{\nabla} V$ where V is electric potential.

$\therefore -\vec{\nabla} V = x\hat{i} - y\hat{j} + 2\hat{k}$

or $-\vec{\nabla} V = -(x\hat{i} + y\hat{j} - 2\hat{k})$

This will satisfy Laplace eqn. if $\nabla^2 V = 0$ or $\vec{\nabla} \cdot \vec{\nabla} V = 0$

Now $\vec{\nabla} \cdot \vec{\nabla} V = \vec{\nabla}[-(x\hat{i} - y\hat{j} + 2\hat{k})]$

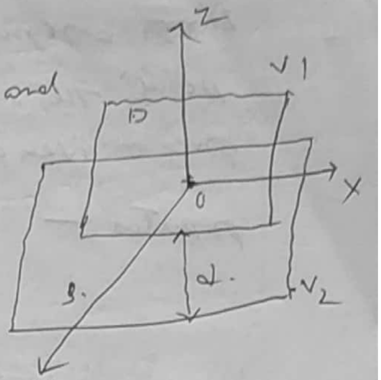
$= -\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (x\hat{i} - y\hat{j} + 2\hat{k})$

$= -[1 - 1 + 0] = 0$

As $\nabla^2 V = 0$, thus satisfies the Laplace's eqn.

CAC

Q Using Laplace's eqn, show that the electric field is constant in the region between the two parallel plates and it is toward the plate of lower potential.



Ans→ Let us consider the parallel plates P and Q. with potential V_1 and V_2 respectively.

These two plates are in x-y plane and distance between them is d. in the z direction.

potential at a point between the plates is given by Laplace's equation $\nabla^2 V = 0$

In the Cartesian Co-ordinate the Laplace's eqn is reduced to $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$ ——— (1)

V is not vary with x and y, therefore equation (1) becomes $\nabla^2 V = \frac{\partial^2 V}{\partial z^2} = 0$ → (2)

$\therefore V = AZ + B$ where A and B are two arbitrary constant.

Using boundary condition

$V = V_1,\ Z = 0,\quad V = V_2\ \ Z = d$

From (2)

$\therefore V_1 = A \times 0 + B$

$\Rightarrow V_1 = B$

and $V_2 = Ad + B$

$\therefore V_2 - V_1 = Ad + B - B = Ad.$

$\Rightarrow A = \frac{V_2 - V_1}{d}$

$\therefore (2) \Rightarrow V = \left(\frac{V_2 - V_1}{d}\right) z + V_1 \quad \{\because V_1 = B\}$

$\therefore$ Electric field

$E = -\left(\frac{\partial V}{\partial z}\hat{z}\right) = -\frac{\partial}{\partial z}\left(\left(\frac{V_2 - V_1}{d}\right) z + V_1\right)\hat{z}$

$= -\frac{(V_2 - V_1)}{d}\hat{z} = \frac{V_1 - V_2}{d}\hat{z}$

= Constant.