Tensor of schaum's outline

(3) Co-ordinate Transformation

Let for two reference frame of the co-ordinate set a be

$$\bar{x}^1 = \bar{x}^1(x^1, x^2 \ldots x^N)$$
$$\bar{x}^2 = \bar{x}^2(x^1, x^2 \ldots x^N)$$
$$\ldots$$
$$\bar{x}^N = \bar{x}^N(x^1, x^2 \ldots x^N)$$
— (1)

by briefly.

$$\bar{x}^K = \bar{x}^K(x^1, x^2 \ldots x^N) \quad K = 1, 2, \ldots N \quad \text{— (2)}$$

For singled-valued, continuous derivation. The conversely be written as

$$x^K = x^K(\bar{x}^1, \bar{x}^2 \ldots \bar{x}^N), \quad K = 1, 2 \ldots N \quad \text{— (3)}$$

he relation (2) or (3) define a transformation of ordinates from one frame to another.

(4) Contravariant and Co-variant vector

Let N quantities $A^1, A^2 \ldots A^N$ in $(x^1, x^2 \ldots x^N)$ system and $\bar{A}^1, \bar{A}^2 \ldots \bar{A}^N$ of $(\bar{x}^1, \bar{x}^2 \ldots \bar{x}^N)$ system. then by transformation eqns in summarised method

$$\bar{A}^p = \sum_{q=1}^{N} \frac{\partial \bar{x}^p}{\partial x^q} A^q \quad , \text{ where } p = 1, 2 \ldots N$$

Conventionally written as

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^q} A^q$$

This is the components of a contravariant vector or the 1st rank or first order contravariant tensor (see Problem 7.33 & 7.34).

on otherwise $A_1, A_2 \ldots A_N$ of N quantities of system $(x^1, x^2 \ldots x^N)$ and $\bar{A}_1, \bar{A}_2 \ldots \bar{A}_N$ of N quantities in $(\bar{x}^1, \bar{x}^2 \ldots \bar{x}^N)$ system. then transformation equations

$$\bar{A}_p = \sum_{q=1}^{N} \frac{\partial x^q}{\partial \bar{x}^p} A_q \quad , \quad p = 1, 2 \ldots N$$

simply written as

$$\bar{A}_p = \frac{\partial x^q}{\partial \bar{x}^p} A_q$$

known: covariant vector or ~~con~~ covariant tensor of the first rank or 1st order.

(5) Contravariant, Covariant, and Mixed Tensor :-

Let N^2 quantities A^{qs} in $(x^1, x^2 .. x^N)$ system & $\bar{A}^{qs}$ in $(\bar{x}^1, \bar{x}^2 ... \bar{x}^N)$ system

$$\bar{A}^{pr} = \sum_{s=1}^{N} \sum_{q=1}^{N} \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} A^{qs}$$

Simply - $\bar{A}^{pr} = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} A^{qs}$

Known as ~~Tens~~ Contravariant tensor of 2nd ~~ran~~ rank or rank two

Mly $\bar{A}_{pr} = \frac{\partial x^q}{\partial \bar{x}^p} \frac{\partial x^s}{\partial \bar{x}^r} A_{qs}$

Covariant tensor of 2nd rank.

&

For Mixed Tensor:

$$\bar{A}^p_r = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial x^s}{\partial \bar{x}^r} A^q_s$$

For 3rd $\bar{A}^{pr}_s = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^n} \frac{\partial x^s}{\partial \bar{x}^t} A^{qn}_t$, 3rd rank

(6) Kronecker Delta

$$\delta^j_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

(7) Tensor of Rank greater than two. ~~Tensor~~

Any mixed tensor A^{qst}_{kl} $\bar{A}^{p}_{ij}$

A^{qst}_{kl} → Rank — 5

Contravariant - order — 3

Covariant order — 2

Can be transform are to the relation

$$\bar{A}^{p r m}_{ij} = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} \frac{\partial \bar{x}^m}{\partial x^t} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} A^{qst}_{kl}$$

(8) Scalar or Invariant

$\phi = \bar{\phi}$ A . scalar or invarion known as

Tensor of rank zero.

(9) Tensor field :- If any N-dimensional space define tensor any vector or scalar ; It called be noted that a tensor or tensor field is not just the set of its components in one spacial. Cordinate system but all the possible sets under any ~~two~~ transformation of co-ordinate.

(10) <u>Symmetric Tensor</u> :-

No change due to two covariant indices or contravariant indices as →

$$A^{mpr}_{qs} = A^{pmr}_{qs} \quad \text{or} \quad A^{mpr}_{qs} = A^{mpr}_{sq}$$

All are symmetric Tensors.

(11) Anti symmetric Tensor

$$A^{mpr}_{qs} = - A^{pmr}_{qs} \quad \text{or.}$$

(12) <u>Operation apply</u>

(a) Addition
$$C^{mp}_{q} = A^{mp}_{q} + B^{mp}_{q}$$
(Commutative & Associative)

Same rank and type

(b) Substraction
$$D^{mp}_{q} = A^{mp}_{q} - B^{mp}_{q}$$

(c) <u>Outer Multiplication</u>
$$A^{pr}_{q} B^{m}_{s} = C^{prm}_{qs}$$

Summation of contravariant & Summation covariant

(14) Inner Product :-

As outer product

$A^{mp}_{q} B^{r}_{st}$

putting $q = r$, $A^{mp}_{r} B^{r}_{st}$, putting $p = s$

another product

$A^{mp}_{r} B^{r}_{pt}$ another inner product is

obtained.

(F) Quotient law :- Suppose it is not known whether a quantity X is a tensor or not. If an inner product of X with an arbitrary tensor is itself a tensor, then X is also a tensor. This is known as quotient law.

(B) Matrix.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Compact form $[a_{pq}]$, $p = 1 \cdots m$, $q = 1 \cdots$

(14) Line Element and Metric Tensor :

Line element

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$= \sum_{p=1}^{3} \sum_{q=1}^{3} g_{pq}\, du_p\, du_q$$

(Curvilinear co-ord)

Known as Three 3D Euclidean space

generally for N - dimension

$$ds^2 = \sum_{p=1}^{N} \sum_{q=1}^{N} g_{pq}\, du_p du_q$$

Conventionally written as

$$ds = g_{pq}\, du_p\, du_q$$

Transformation of co-ordinate from x^j to $\bar{x}^k$

such that ~~space~~ the metric form ~~into~~

transformation into

$$(d\bar{x}^1)^2 + (d\bar{x}^2)^2 + \dots + (d\bar{x}^N)^2 \text{ or}$$

$$d\bar{x}^k\, d\bar{x}^k$$

This type coordinate called

N- dimensional Euclidean space; in general known as "Riemanian".

(15) <u>Conjugate or Reciprocal Tensor</u>

Let $g = |g_{pq}|$ denote the determinant with elements g_{pq} and suppose $g \neq 0$, $g^{ij} = \frac{1}{g}$ defined by g^{pq} as

$$g^{pq} = \frac{\text{Co-factor of } g_{pq}}{g}$$

This g^{pq} is a symmetric contravariant Tensor of rank Two ~~or~~ called the conjugate or reciprocal Tensor of g_{pq}. valid for

$$g^{pq} g_{rq} = \delta^p_r \quad //$$

(16) Associated Tensors

All tensors obtained from a given tensor by forming inner products with the metric tensor and its conjugate are called associate tensors of the given tensor.

eg. A^m and A_m are associated. tensor, the first are contravariant & 2nd covariant components. The relation between them is given by

$$A_p = g_{pq}A^q \quad \text{or} \quad A^p = g^{pq}A_q$$

For ~~an~~ Cartesian coordinate system

$$g_{pq} = 1 \text{ if } p = q$$
$$= 0 \text{ if } p \neq q$$

so that $A_p = A^p$, no difference between contravariant & covariant components.

(17) Christoffel's Symbols :—

The following symbols

$$[pq, r] = \frac{1}{2}\left(\frac{\partial g_{pr}}{\partial x^q} + \frac{\partial g_{qr}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^r} \right)$$

$$\left\{ {s \atop pq} \right\} = g^{sr} [pq, r]$$

Known as Christoffel's Symbols of the first and 2nd kind. The 2nd kind $\left\{ {s \atop pq} \right\}$ are $\{pq, s\}$ and Γ^{s}_{pq}

(18) Transformation laws of Christoffel symbols

Suppose for co-ordinate $\bar{x}^k$ then

$$\overline{[jk, m]} = [pq, r] \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} \frac{\partial x^r}{\partial \bar{x}^m} + g_{pq} \frac{\partial x^p}{\partial \bar{x}^m} \cdot \frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^k}$$

$$\overline{\left\{ {n \atop jk} \right\}} = \left\{ {s \atop pq} \right\} \frac{\partial \bar{x}^n}{\partial x^s} \cdot \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} + \frac{\partial \bar{x}^n}{\partial x^q} \frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^k}$$

are the laws of transformation of the Christoffel symbols showing that they are not tensors unless the terms on the right are zero

Meaning of Christoffel

However, since g_{ik} is a tensor, its covariant derivatives are & can be expressed in terms of regular partial derivatives and Christoffel symbols.

$$g_{ik,l} = \frac{\partial g_{ik}}{\partial x^l} - g_{mk}\Gamma^m_{il} - g_{im}\Gamma^m_{kl} = 0$$

Rewriting the above expression and then performing permutation of i, k and l

$$\frac{\partial g_{ik}}{\partial x^l} = g_{mk}\Gamma^m_{il} + g_{im}\Gamma^m_{kl}$$

$$\frac{\partial g_{kl}}{\partial x^i} = g_{ml}\Gamma^m_{ki} + g_{km}\Gamma^m_{li}$$

$$-\frac{\partial g_{li}}{\partial x^k} = -g_{mi}\Gamma^m_{lk} - g_{ml}\Gamma^m_{ik}$$

Adding these eqns -

$$\frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{kl}}{\partial x^i} - \frac{\partial g_{li}}{\partial x^k} =$$

$2g_{mk}\Gamma^{m}_{il}$.

$\therefore 2g_{mk}\Gamma^{m}_{il} = g_{ik,l} + g_{kl,i} - g_{li,k}$

Rate of change space

Euclidean Matrix

$(ds)^2 = (dx^i)^2, \quad i = 1,2,3$

$= (dx^1)^2 + (dx^2)^2 + (dx^3)^2$

Reimann matrix

$ds^2 = g_{ij}\, dx^i dx^j, \quad g_{ij} = g_{ji}$

$$\begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = g_{ij}$$