8. Remainder term in Newton's forward Interpolation formula:

Let $P_n(x)$ be a pollynomial of the n-th degree in x representing the given function $y = f(x)$. We assume that the function $f(x)$ possesses a continuous $(n+1)$ order derivatives in the interval of interpolation $x_0 \le x \le x_n$. Since $P_n(x)$ has the same value as $f(x)$ at the points $x_0, x_1, x_2, \cdots, x_n$

$$\therefore f(x) = P_n(x) + g(x) \longrightarrow (i)$$

where $g(x)$ has the root $x_0, x_1, x_2, \cdots, x_n$.

$$\therefore f(x) = P_n(x) + k(x)\left[(x-x_0)(x-x_1)\cdots(x-x_n)\right] \longrightarrow (ii)$$

where $k(x)$ is the function to be determined in terms of the function $f(x)$. This can be done as follows:

We construct the function

$$\cancel{\psi(t) = f(t) - P_n(t) - k(x)[(x-x_0)(x-x_1)\cdots(t-x_n)]}$$

$$\psi(t) = f(t) - P_n(t) - k(x)\left[(t-x_0)(t-x_1)\cdots(t-x_n)\right] \longrightarrow (iii)$$

Now by virtue of (ii) this equation has at least $(n+2)$ real roots $(t = x, x_0, x_1, \cdots, x_n)$.

Therefore by Rolle's Theorem $\psi'(t)$ has at least $(n+1)$ real roots lying between the smallest and the greatest of the above roots. Similarly $\psi''(t)$ has at least n real roots in this interval and finally the $(n+1)$-th derivative of $\psi(t)$ i.e. $\psi^{n+1}(t)$ has at least one root $t = \xi$ (say) in the interval x_0 to x_n.

Taking the $(n+1)$-th derivative of each side of (iii), we have

$$\psi^{(n+1)}(t) = f^{n+1}(t) - P_n^{(n+1)}(t) - k(x)\,\lfloor(n+1) \longrightarrow (iv)$$

Since $P_n^{n+1}(x) = 0$ and ξ is a root of L.H.S. of (iv),

(because $P_n(x)$ is a polly of n-th degree)

Therefore

$$\psi^{n+1}(t) = 0 \quad at \quad t = \xi .$$

Hence (iv) $\Rightarrow f^{n+1}(\xi) - k(x)\,(n+1)! = 0$

$$\Rightarrow k(x) = \frac{1}{(n+1)!} f^{n+1}(\xi), \quad x_0 \leq \xi < x_n .$$

Substituting $k(x)$ is (ii), we have

$$f(x) = P_n(x) + \frac{1}{\lfloor n+1} f^{n+1}(\xi)\left[(x-x_0)(x-x_1)\cdots(x-x_n)\right]$$

or, $f(x) - P_n(x) = \frac{1}{\lfloor n+1} f^{n+1}(\xi)\left\{(x-x_0)(x-x_1)\cdots(x-x_n)\right\}$

or, $R_n = \frac{1}{\lfloor n+1} f^{n+1}(\xi)\left[(x-x_0)(x-x_1)\cdots(x-x_n)\right]$

where $R_n = f(x) - P_n(x)$.

or, $\therefore R_n = \frac{f^{n+1}(\xi)}{\lfloor n+1} \prod_{i=0}^{n} (x-x_i) \longrightarrow$ (V)

Substituting $\frac{x-x_0}{h} = u$ i.e. $x = x_0 + hu$, $\frac{x-x_1}{h}$ etc in (V), we have,

$\left[\frac{x-x_0}{h} = u,\ x_1 - x_0 = h \Rightarrow x_1 = x_0 + h,\ \text{|||}^{ly}\ x_2 = x_0 + 2h \text{ etc}\right]$

$$R_n = \frac{u(u-1)(u-2)\cdots(u-n)}{\lfloor (n+1)} h^{n+1} f^{n+1}(\xi), \quad x_0 < \xi < x_n .$$

R_n is called remainder term (or truncation error) in Newton's forward interpolation formula.

Hence Newton's forward interpolation formula with remainder term is

$$f(x) = y_0 + u\Delta y_0 + u(u-1)\frac{\Delta^2 y_0}{2!} + u(u-1)(u-2)\frac{\Delta^3 y_0}{3!} + \cdots$$

$$+ \cdots + u(u-1)(u-2)\cdots(u-n+1)\frac{\Delta^n y_0}{n!} + \frac{u(u-1)(u-2)\cdots(u-n)\,h^{n+1} f^{n+1}(\xi)}{\lfloor n+1}$$

Derive Newton's forward interpolation formula with remainder term. Marks 8

[Ans: Derive in terms of x only].

Remainder term in Newton's backward interpolation formula:

Let $P_n(x)$ be a polynomial of nth degree in x representing the given function $y = f(x)$. We assume that $f(x)$ posses continuous $(n+1)$-th order derivatives in the interval of interpolation, $x_0 \le x \le x_n$. Since $P_n(x)$ has the same values of the points $x_0, x_1, \ldots, x_n$; therefore we can write

$$f(x) = P_n(x) + g(x) \longrightarrow (i)$$

where $g(x)$ has the roots $x_n, x_{n-1}, \ldots, x_1, x_0$.

$$\therefore f(x) = P_n(x) + k(x)\left[(x-x_n)(x-x_{n-1})\cdots(x-x_0)\right] \longrightarrow (ii)$$

where $k(x)$ is to be determined in terms of the function $f(x)$. This can be done as follows:

We consider the function

$$\psi(t) = f(t) - P_n(t) - k(x)\left[(t-x_n)(t-x_{n-1})\cdots(t-x_0)\right] \longrightarrow (iii)$$

Now, by virtue of (ii), this equation (iii) has at least $(n+2)$ real roots $x_n, x_{n-1}, \ldots, x_1, x_0, x$. Therefore by Rolle's theorem $\psi'(t)$ has at least $(n+1)$ real roots lying between smallest and greatest value of the above roots. Thus the $(n+1)$ th derivative $\psi^{n+1}(t)$ has at least one real root $t = \xi$ in the interval x_0 to x_n.

Taking the $(n+1)$-th derivatives of ~~the~~ both sides of (iii), we get

$$\psi^{n+1}(t) = f^{n+1}(t) - k(x).(n+1)! \quad [\because P^{n+1}(t) = 0]$$

Again, since $\psi^{n+1}(t) = 0$ for $t = \xi$, $x_0 < \xi < x_n$,

$$\therefore f^{n+1}(\xi) - k(x).(n+1)! = 0$$

$$\Rightarrow k(x) = \frac{f^{n+1}(\xi)}{(n+1)!}$$

Substituting this value of $k(x)$ in (ii), we get

$$f_n(x) - P_n(x) = \frac{f^{n+1}(\xi)}{(n+1)!}(x-x_n)(x-x_{n-1})\cdots(x-x_0)$$

or, $$R_n = \frac{f^{n+1}(\xi)}{(n+1)!}(x-x_n)(x-x_{n-1})\cdots(x-x_0) \rightarrow \text{(iv)}$$

Now putting $\frac{x-x_n}{h} = u$, $\frac{x-x_{n-1}}{h} = u+1$, ------

------, $\frac{x-x_0}{h} = u+n$ etc. in (iv), we have

$$\text{Error} = R_n = \frac{f^{n+1}(\xi)}{(n+1)!} h^{n+1} u(u+1)(u+2)\cdots(u+n).$$

This is the remainder term in Newton's backward interpolation formulae.

Remainder term in Newton's general interpolation formula for unequal intervals :→

Let $P_n(x)$ be a polynomial of n-th degree in x representing the given function $y = f(x)$. Then the equation (vi) can be written in the form

$$f(x) = P_n(x) + R_n(x).$$

We assume that the $(n+1)$-th derivative of $f(x)$ exists and is continuous, the function $\delta(x, x_0, x_1, \ldots, x_n)$ is finite. Therefore by (vii) R_n vanishes at $(n+1)$ points $x_0, x_1, x_2, \ldots, x_n$. Thus by Rolle's theorem, $R_n'(x)$ vanishes at n points lying between the ~~sm~~ smallest and the largest of those arguments and so on. Finally, n-th derivative of $R_n(x)$ vanishes for some point $x = \xi$, therefore

$$f^n(\xi) = n!\, \delta(x_0, x_1, x_2, \ldots, x_n), \quad x_0 < \xi < x_n$$

$$\therefore \delta(x_0, x_1, x_2, \ldots, x_n) = \frac{1}{n!} f^n(\xi)$$

$$\therefore \delta(x, x_0, x_1, x_2, \ldots, x_n) = \frac{1}{(n+1)!} f^{n+1}(\xi)$$

$$\therefore R_n = \frac{1}{(n+1)!} f^{n+1}(\xi) \cdot (x - x_0)(x - x_1)(x - x_2) \cdots (x - x_n)$$

$$= \frac{f^{n+1}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i).$$

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Remainder term in Lagrange's formula:

To find the remainder term in Lagrange's formula we construct the function

$$F(t) = f(t) - \phi(t) - [f(x) - \phi(x)] \frac{(t-x_0)(t-x_1)\cdots(t-x_n)}{(x-x_0)(x-x_1)\cdots(x-x_n)} \rightarrow (1)$$

where $f(x)$ denotes the given function, $\phi(x)$ a polynomial interpolation formula ($\phi(x)$ is a poly of degree n), and 't' is a real variable.

Let $f(x)$ is continuous and posses continuous derivatives upto $(n+1)^{th}$ ~~der~~ order.

Now, $F(t)$ vanishes for $(n+2)$ values of t i.e. $t = x, x_0, x_1, \ldots, x_n, t$. and since $f(x)$ is continuous & it has continuous derivatives upto $(n+1)$-th order. Therefore the same is ~~&~~ true for $f(t)$ and of $F(t)$. $F(t)$ thus satisfies the condition of Rolle's theorem. Hence the $(n+1)$-th derivative of $F(t)$ will vanish at least once at some point ξ with the interval from x_0 to x_n.

Differentiating (i) $(n+1)$ times w.r.t. 't' we have,

$$F^{n+1}(t) = f^{n+1}(t) - \phi^{n+1}(t) - [f(x) - \phi(x)] \cdot \frac{(n+1)!}{(x-x_0)(x-x_1)\cdots(x-x_n)}$$

or, $$F^{n+1}(t) = f^{n+1}(t) - [f(x) - \phi(x)] \frac{(n+1)!}{(x-x_0)(x-x_1)\cdots(x-x_n)} \rightarrow (2)$$

But $F^{n+1}(t) = 0$ for some $t = \xi$, $x_0 < \xi < x_n$.

Therefore we have from (2),

$$f^{n+1}(\xi) = [f(x) - \phi(x)] \frac{(n+1)!}{(x-x_0)(x-x_1)\cdots(x-x_n)}$$

$$\Rightarrow f(x) - \phi(x) = \frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n).$$

i.e. $$R_n = \frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n),$$

$$= \frac{f^{n+1}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x-x_i) \qquad x_0 < \xi < x_n.$$

which is the remainder term in Lagrange's interpolation formula for unequal intervals.

Note: This R_n is same with the R_n in Newton's general Int. formula