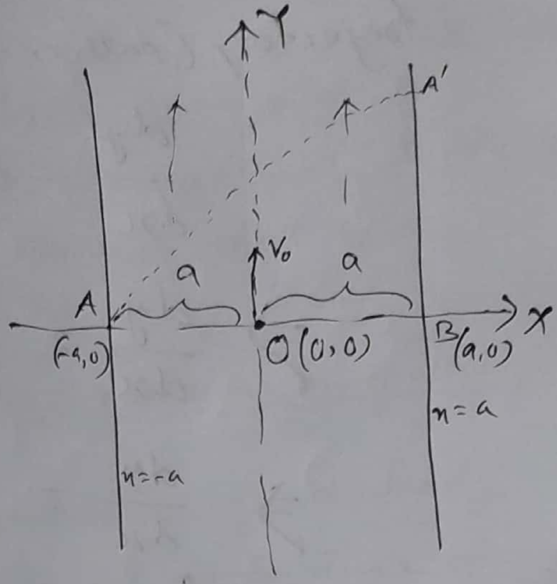
Swimmer's Problem :-

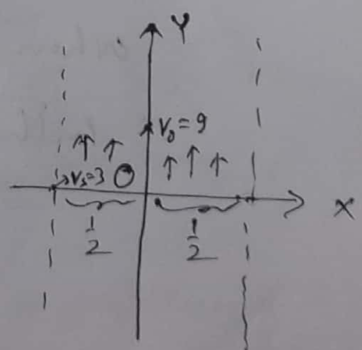


We consider a northward river of width $w=2a$. The lines $x=\pm a$ represents the banks of the river and y axis its centre. If the swimmer starts at the point $A(-a,0)$ on the west bank and swims towards east with constant speed V_s. Then the differential equation

$$\frac{dy}{dx} = \frac{V_0}{V_s}\left(1-\frac{x^2}{a^2}\right) \longrightarrow (1)$$

gives the swimmer's trajectory $y=y(x)$ as he crosses the river. Here V_0 is the midstream velocity, x is the distence from the centre of the river, V_s is the velocity of the swimmer.

Example 1. Suppose a 1 mile wide river has 9 miles/hour midstream velocity. A swimmer is swimming with a velocity of 3 miles/hour. Find how far downstream the swimmer drifts as he crosses the river.



Solution :- Here, the width of the river $=2a=1$.

$$\left.\begin{aligned} &\Rightarrow a=\tfrac{1}{2} \\ &\text{Also, here, } V_0 = 9 \\ &\qquad\qquad V_s = 3. \end{aligned}\right\} \rightarrow (1)$$

Now, we know that the diff. eqn. of the trajectory (path or curve) of the swimmer is

$$\frac{dy}{dx} = \frac{V_0}{V_s}\left(1 - \frac{x^2}{a^2}\right) \longrightarrow (2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{9}{3}\left(1 - \frac{x^2}{(1/2)^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = 3(1-4x^2)$$

$$\Rightarrow \int dy = 3\int(1-4x^2)\,dx$$

$$\Rightarrow y = 3(x - 4x^3/3) + c_1$$

$$= 3x - 4x^3 + c_1 \longrightarrow (3)$$

Since the swimmer starts at $x = -\frac{1}{2}$, so $y(-\frac{1}{2}) = 0$. So putting $x = -\frac{1}{2}$ and $y = 0$ in (3),

$$c_1 = 4\cdot\left(\frac{1}{-2}\right)^3 - 3\cdot\left(-\frac{1}{2}\right)$$

$$= \frac{-4}{8} + \frac{3}{2} = \frac{-4+12}{8} = \frac{8}{8} = 1$$

$\therefore$ (3) Reduces to

$$y = 3x - 4x^3 + 1 \longrightarrow (4)$$

which is the trajectory of the swimmer.

When the swimer cross the river, then $x = \frac{1}{2}$.

$\therefore$ putting $x = \frac{1}{2}$ in (4),

$$y = 3\times\frac{1}{2} - 4\cdot\left(\frac{1}{2}\right)^3 + 1$$

$$= \frac{3}{2} - \frac{4}{8} + 1$$

$$= \frac{12-4+8}{8} = \frac{16}{8} = 2.$$

So, the swimmer drifts 2 miles downstream while he swims 1 miles across the river. #

Newton's Law of Cooling :→

Statement: The rate of change of temperature of a body is proportional to the difference in temperature between that of surrounding and that of the body itself :

Mathematical modelling of Newton's law of cooling:

Let T be the temperature of a body at any time t. Let S be the temperature of the surroundings.

$\therefore$ rate of change of temperature $= \dfrac{dT}{dt}$

$\therefore$ By Newton's law of cooling

$$\frac{dT}{dt} \propto (T - S)$$

$$\Rightarrow \frac{dT}{dt} = -\lambda(T - S), \text{ where } \lambda \text{ is a positive const.} \longrightarrow (i)$$

which is the diff. eqn. (mathematical model of Newton's law of cooling).

Case (i): If $T > S$, then $\dfrac{dT}{dt} < 0$.

This means that the temp^r is a decreasing fn. of t and the body is cooling.

Case (ii): If $T < S$, then $\dfrac{dT}{dt} > 0$. This means that the temperature is an increasing fn. of t and the body is heating.

Note :- If we are given the values of λ and S, we should be able to find $T(t)$ which can be helpful in predicting the future temperature of the body.

Example: A pitcher of buttermilk initially at 25°C is to be cooled by setting it on the front porch, where the temperature is 0°C. Suppose that the temperature of the buttermilk has dropped to 15°C after 20 minutes. When will it be at 5°C ?

Solⁿ. Given

at $t=0$, $T=25°C$

at $t=20$, $T=15°$

Temperature at surroundings $= S = 0°C$

We will find t when $T=5°C$.

By Newton's law of cooling, we know

$$\frac{dT}{dt} = -\lambda(T-S),$$

$$\Rightarrow \frac{dT}{T-S} = -\lambda\, dt$$

$$\Rightarrow \int \frac{dT}{T-S} = \int(-\lambda)\, dt$$

$$\Rightarrow \log(T-S) = -\lambda t + \log c$$

$$\Rightarrow T-S = ce^{-\lambda t} \longrightarrow (1)$$

We put $t=0$, $T=25$ and $S=0$ in (1). So we get

$$25-0 = c\, e^{-\lambda(0)} \Rightarrow c = 25$$

$$\therefore T = 25e^{-\lambda t} \longrightarrow (2)$$

Now, putting $t=20$, $T=15$ in (2), we get

$$15 = 25e^{-\lambda 20} \Rightarrow e^{-20\lambda} = \frac{15}{25} = 0.6$$

$$\Rightarrow \log_e(0.6) = -20\lambda$$

$$\Rightarrow \log_{10}(0.6)\times\log_e(10) = -20\lambda$$

$$\Rightarrow \log_{10}(0.6) = -20\lambda\log_{10}(e)$$

$$\Rightarrow \bar{1}.7782 = -20\lambda(0.4343)$$

$$\Rightarrow -0.2218 = -8.686\lambda$$

$$\Rightarrow \lambda = 0.0255 \cong 0.026$$

$e = 2.718281$

$$\begin{array}{r} 4330 \\ +13 \\ \hline 4343 \end{array}$$

Putting $\lambda = 0.026$ in (2) we get

$$T = 25\, e^{-(0.026)t} \rightarrow (3)$$

Putting $T = 5$ in (3), we get -

$$5 = 25\, e^{-(0.026)t}$$

$$\Rightarrow \frac{1}{5} = e^{-0.026t}$$

$$\Rightarrow \log_e(1/5) = -0.026t$$

$$\Rightarrow -\log_e 5 = -0.026t$$

$$\Rightarrow +\log_{10} 5 \times \log_e 10 = +0.026t$$

$$\Rightarrow \log_{10} 5 = 0.026t \times \log_{10}(e)$$

$$= 0.026t \times (0.4343)$$

$$\Rightarrow 0.6990 = (0.0112918)t$$

$$t = 61.9033$$

$\therefore$ Required time = 61.90 mins.

Growth & Decay :

Natural Growth eqn: The diff. eqn.

$$\frac{dx}{dt} = \mu(x), \quad x(t) > 0, \; \mu > 0$$

is called a natural growth eqn. or exponential eqn.

Natural Decay eqn: The diff. eqn

$$\frac{dx}{dt} = \mu(x), \quad x(t) > 0, \; \mu < 0$$

is called a natural decay eqn.

Population Growth:

Let $P(t)$ be the population having constant birth and death rates. Then the time rate of change of population $P(t)$ is proportional to the size of the population. So, we have

$$\frac{dP}{dt} = \lambda P \quad \text{, where } \lambda \text{ is const. of proportionality} \longrightarrow (1)$$

which is the diff. eqn. of population growth.

$$(1) \Rightarrow \frac{dP}{P} = \lambda \, dt$$

$$\Rightarrow \int \frac{dP}{P} = \lambda \int dt$$

$$\Rightarrow \log_e P = \lambda t + c \longrightarrow (2)$$

which is the general solⁿ of the population growth model (1).

Let the initial population be P_0.

$\therefore$ We put $P = P_0$ when $t = 0$. So (2) gives

$$\log_e P_0 = \lambda(0) + c \Rightarrow c = \log_e P_0$$

$\therefore$ (2) reduces to

$$\log_e P = \lambda t + \log_e P_0$$

$$\Rightarrow \log_e P - \log_e P_0 = \lambda t$$

$$\Rightarrow \log_e \left(\frac{P}{P_0}\right) = \lambda t$$

$$\Rightarrow \frac{P}{P_0} = e^{\lambda t}$$

$$\Rightarrow P = P_0 e^{\lambda t} \longrightarrow (3)$$

which is the population at any time t if the initial population is P_0.