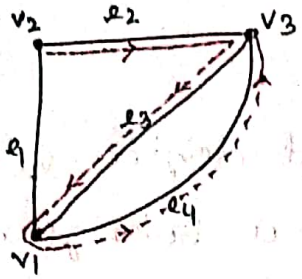
2015

① Determine if the walk $(v_2, e_2, v_3, e_3, v_1, e_4, v_3)$ is a path. (1)



It is not a path since the walk has a repeated vertex v_3.

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② Does ∃ a simple graph with five vertices having degrees 2, 2, 4, 4, 4 ? Justify.

Soln: There ∄ any simple graph satisfying the given properties. Suppose there exists a simple graph with five vertices v_1, v_2, v_3, v_4, v_5 having degrees 2, 2, 4, 4, 4 respectively. Since the graph does not contain any loop and any parallel edges, it follows that each of the vertices v_3, v_4, v_5 must have four adjacent vertices. Hence v_1 is adjacent vertex of v_3, v_4, v_5. This implies that the degree of v_1 is at least 3, which goes against the assumption. Hence there is no such graph.

③ How many vertices are there in a graph with 15 edges if each vertex is of degree 3?

Soln: Let there be n vertices. Then the sum of the degrees of the vertices is $3n$. This sum is twice the no. of edges.

Hence $3n = 30$ i.e. $n = 10$.

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④ If a graph G_1 contains exactly two vertices of odd degree, show that there exists a path between these two vertices.

Pf: Let u and v be the odd degree vertices of G_1. Consider the component C to which u belongs. Now C is a connected subgraph of G_1. We know that the no. of odd degree vertices in a graph is even. Since u and v are the only odd degree vertices in G_1, it follows that $v \in C$. Hence there exists a path from u to v.

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⑤ Prove that a connected graph T with n vertices is a tree iff T contains $(n-1)$ edges.

Pf: Let T be a tree with n vertices. We prove the th^m by induction on n.

First, let $n=1$. Since a tree, it has no cycle. Hence T, with only one vertex, can not have any edge, i.e no. of edges of $T = 0$. Thus $n-1 = 1-1 = 0$. So the th^m is true for $n=1$

Next, we assume that the th^m is true for any tree with k vertices. Let T be a tree with $k+1$ vertices where $k+1 \geq 2$. Then T has a vertex, say u, of degree 1. We construct a new graph T^* by removing from T the vertex u and the edge incident with it.

Since T is connected having no cycle, T^* is also connected and contains no cycle. Hence T^* is a tree with k vertices. By induction hypothesis, T^* has $k-1$ edges and hence T has $(k-1)+1 = k$ edges.

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Hence, by principle of induction, a tree with n vertices has n-1 edges.

Conversely, let T be a connected graph with n vertices and n-1 edges. We show that T is a tree.

If possible, let T contains a cycle C. If we delete an edge from C we get a connected graph with n vertices and n-2 edges. This contradicts the fact that a connected graph with n vertices contains at least n-1 edges. Hence T does not contain any cycle, i.e T is a tree.

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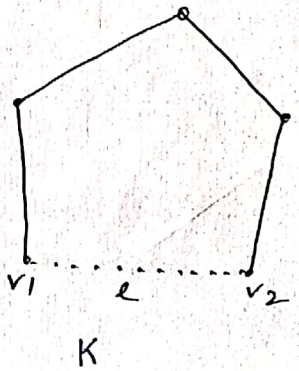
⑥ Prove that a connected graph G remains connected after removing an edge e from G iff e is in some cycle in G. (Alternatively, let e be an edge of a connected graph G. Then G-e is connected iff e is a cycle-edge of G).

Pf: Let G be a connected graph and e be an edge contained in some cycle in G. We may assume that the edge is not a loop or a parallel edge, because



K

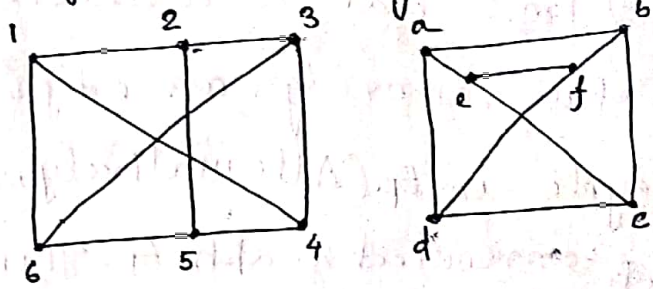
deletion of such an edge does not change the property of connectedness. We consider a cycle K as shown in fig:

Let the edge e joining the vertices v_1 and v_2 in the cycle K be deleted from G. Along e we may go from v_1 to v_2 (or from v_2 to v_1). But we can also reach v_2 from v_1 (or from v_1 to v_2) by going along the other edges of K in the graph G-e.

Thus there is a path between this pair of vertices and hence bet^n any pair of vertices. Hence the new graph G-e is also connected.

conversely, let G-e be connected. Then there is a path P from v_1 to v_2 which avoids e. Thus P+e is a cycle containing e ie e is a cycle-edge.

(7) Examine if the foll. two graphs display an isomorphism bet^n them



Sol^n:

1 - a
2 - b
3 - e
4 - d
5 - c
6 - f

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⑧ P/T there is always a Hamiltonian path in a directed complete graph.

Sol^n: Let there be a length $p-1$ ($p<n$) path, $(v_1, v_2 \ldots, v_p)$ Let v_x be a vertex that is not included in this path, and there is no edge from v_x to v_1. However $(v_1, v_x) \in G$.

Suppose that (v_x, v_2) is also an edge in the path. Replace the edge (v_1, v_2) in the original path with the two edges (v_1, v_x) and (v_x, v_2) so that the vertex v_x will be included in the argument path.

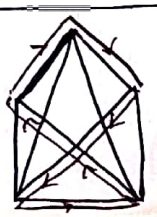
If there is no edge from v_x to v_2, then there must be an edge (v_2, v_x) in the path and we can repeat the argument.

If we find that it is not possible to include vertex v_k in any argument path by replacing an edge (v_k, v_{k+1}) in the original path, with two edges (v_k, v_x) and (v_x, v_{k+1}) with $1 \le k \le p-1$, then we conclude that there must be an edge (v_p, v_x) in the graph.

We can repeat the argument until all the vertices in the graph are included in the argumented path

(OR: We can form a Hamiltonian circuit in complete graph beginning at any vertex. Such a circuit can be built by visiting vertices in order we choose, as long as the path begins and ends at the same vertex and visit each other vertex exactly once. This is possible since there are edges in complete graph between any two vertices.

eg. K_5



Dark Lines shows the Hamiltonian circuit.

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2016

⑨ Can we draw a graph of 7 vertices such that the degree of each vertex is 3? If not, why?

Soln $\sum \deg(v_i) = 7 \times 3 = 21$, odd no.

So ~~#~~ a graph with 7 vertices each of degree 3.

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⑩ Draw the graph $\overline{K_3} + \overline{K_4}$

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⑪ Can a graph containing a cycle of length 3 be a bipartite graph? Justify.

Soln: we know that A graph is bipartite iff it does not contain any cycle of odd length.

So, a graph containing a cycle of length 3 cannot be a bipartite graph.

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(12) Let G be a (p, q) graph, all of whose vertices have degree K or K+1. If G has $p_K > 0$ vertices of degree K and p_{K+1} vertices of degree K+1, then P/T

$$p_K = (K+1)p - 2q$$

Soln sum of the degrees of the vertices is twice the no. of edges so,

$$2q = p_K \times K + p_{K+1} \times (K+1) \longrightarrow (1)$$

Also, $p_K + p_{K+1} = p$

$$\Rightarrow p_{K+1} = p - p_K \longrightarrow (2)$$

(1), (2) $\Rightarrow$

$$2q = p_K \times K + p_{K+1} \times K + p_{K+1}$$

$$= K(p_K + p_{K+1}) + p - p_K$$

$$= Kp + p - p_K$$

$$= (K+1)p$$

$\therefore p_K = (K+1)p - 2q$.

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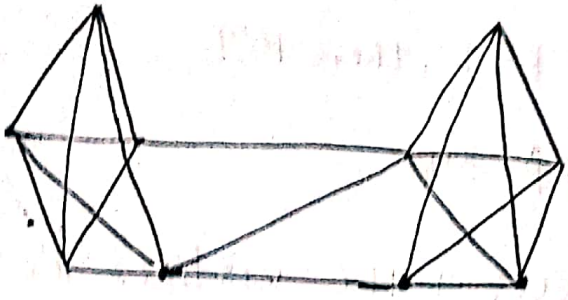
(13) Draw a graph G for which $K(G)=2$, $\lambda(G)=3$, $\delta(G)=4$. (3)



(14) Show that the connectivity fn is a strictly decreasing function (3)

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(15) How many Hamiltonian cycles are there in K_4. (5b)

Soln No. of Hamiltonian cycle are: $\frac{(n-1)!}{2} = \frac{(4-1)!}{2}$

$= \frac{3!}{2} = 3.$

2017

(16) If $R(m,n)$ is a Ramsay number, find $r(2,2)$.

$= 2$ (1)

Result: $R(m,2) = m$ for all m

(17) Let G_1 be a graph with n edges and m vertices, then

$\deg(v_1) + \deg(v_2) + \cdots + \deg(v_m) = 2n$

(18) Let H be a subgraph of a graph G_1 and $\bar{H}$ be the complement of H in G. Draw the graphs H and $\bar{H}$ from the graph G_1.

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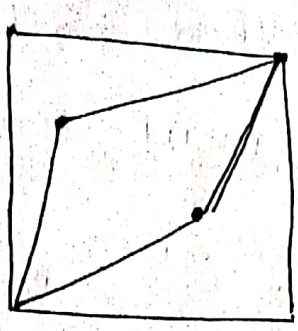
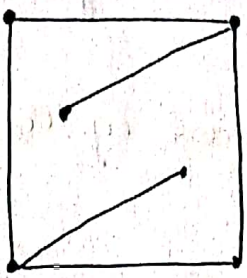


G



H

$\bar{H}$

H Subgraph G

(19) State the hand-shaking theorem. What is meant by source and sink in graph theory? (2)

soln statement: The sum of degrees of the vertices in an undirected graph is even.

**

Source: A vertex with zero in-degree is called a source

Sink: A vertex with zero out-degree is called a sink.

**[The above theorem holds this rule that if several people shake hands, the total no. of hands shake must be even that is why the theorem is called handshaking theorem].

(20) If $G=(V, E)$ is a (p, q) graph, then S/T $\delta(G) \leq \frac{2q}{p} \leq \Delta(G)$ where $\delta(G)$ & $\Delta(G)$ have their usual meanings.

soln: In a graph G with p vertices and q lines, $\frac{2q}{p}$ represents the average degree in the graph. Nevertheless we can prove this inequality algebraically first we know by the Handshaking Lemma that

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$$\sum_{v \in V} \deg(v) = 2q$$

Hence it follows that if $\delta(G)$ is the minimum degree in the graph, then,

$$\sum \delta(G) \leq \sum_{v \in V} \deg(v)$$

$$\Rightarrow p\,\delta(G) \leq 2q$$

$$\Rightarrow \delta(G) \leq \frac{2q}{p}$$

Similarly, if $\Delta(G)$ is the maximum degree in the graph, then,

$$\sum_{v \in V} \deg(v) \leq \sum \Delta(G)$$

$$\Rightarrow 2q \leq p\,\Delta(G)$$

$$\Rightarrow \frac{2q}{p} \leq \Delta(G)$$

$$\therefore \delta(G) \leq \frac{2q}{p} \leq \Delta(G).$$

21) If G is not a connected graph, then S/T $\bar{G}$ is connected. [~~If a graph is not~~ Either G or $\bar{G}$ is connected]

pf: Let G is disconnected. We want to show that

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$\bar{G}$ is connected.

Suppose v and w are two vertices. If vw is not an edge in G, then it is an edge in $\bar{G}$, and so we have a path from v to w in $\bar{G}$. On the other hand, if vw is an edge in G, then this means v and w are in the same component of G. Since G is disconnected, we can find a vertex u in a different component, so that neither uv nor uw are edges of G. Then v-u-w is a path from v to w in $\bar{G}$. This shows that any two vertices in $\bar{G}$ have a path (in fact a path of length one or two) between them in $\bar{G}$. So $\bar{G}$ is connected.

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(22) If G is a simple graph with n vertices and k components, then show that G can have at most $\frac{(n-k)(n-k+1)}{2}$ edges.

Pf: Let G be a simple graph with k components $c_1, c_2, c_3, \cdots, c_k$. Let n_i be the no. of vertices in the component c_i, $i = 1, 2, \cdots, k$. Then

$$\sum_{i=1}^{k} n_i = n.$$

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Since G_i is a simple graph, the maximum possible no. of edges in G_i is $\frac{1}{2} n_i (n_i - 1)$. Hence the no. of edges in G is

$$\leq \sum_{i=1}^{k} \frac{1}{2} n_i (n_i - 1) = \frac{1}{2} \sum_{i=1}^{k} n_i^2 - \frac{1}{2} \sum_{i=1}^{k} n_i$$

$$= \frac{1}{2} \sum_{i=1}^{k} n_i^2 - \frac{1}{2} n.$$

Now for any k positive integers $n_1, n_2, \cdots, n_k$ we have

$$(n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1) = (n_1 + n_2 + \cdots + n_k) - k = n - k.$$

Then, $(n_1 - 1)^2 + (n_2 - 1)^2 + \cdots + (n_k - 1)^2 + 2(n_1 - 1)(n_2 - 1) + 2(n_1 - 1)(n_3 - 1) + \cdots = (n - k)^2$

Hence, $(n_1 - 1)^2 + (n_2 - 1)^2 + \cdots + (n_k - 1)^2 \leq (n - k)^2$

i.e. $n_1^2 + n_2^2 + \cdots + n_k^2 - 2(n_1 + n_2 + \cdots + n_k) + k \leq (n - k)^2$

Hence the no. of edges is

$$\leq \frac{1}{2} \sum_{i=1}^{k} n_i^2 - \frac{1}{2} n \leq \frac{1}{2} \left((n-k)^2 + 2n - k \right) - \frac{1}{2} n$$

$$= \frac{1}{2} \left((n-k)^2 + 2n - k - n \right)$$

$$= \frac{1}{2} \left((n-k)^2 + (n-k) \right)$$

$$= \frac{1}{2} (n-k)(n-k+1).$$

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G is connected. ∎

(23) Define Hamiltonian path and Hamiltonian circuit with an example for each.

(24) Let G be a connected graph which contains an Euler trail, but not Euler circuit, then show that G has exactly two vertices of odd degree.

Pf: Let G be a connected graph with an Euler trail. Let P be an Euler trail from a vertex u to a vertex v of G.

Now construct a new graph G_1 by adding a new edge e to G with u and v as end vertices.

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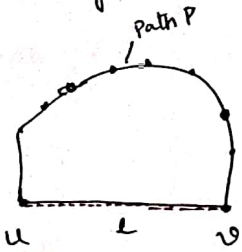
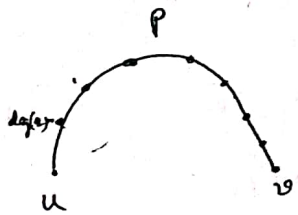
In this graph G_{11}, the tail, P together with e forms an Euler circuit. Hence every vertex of the graph G_{11} is of even degree. This implies that u and v are the only vertices of odd degree.



G_1



$deg(u) = 1$, $deg(v) = 1$

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