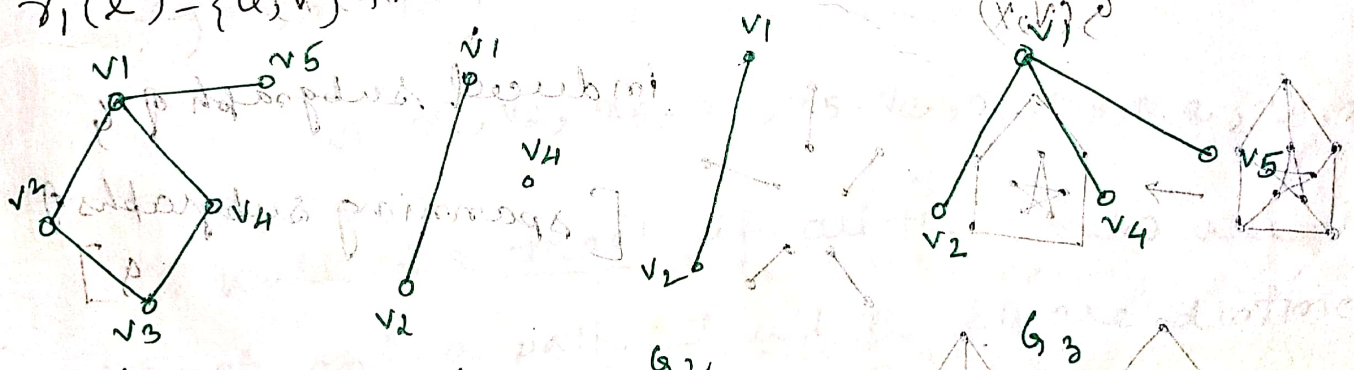
Subgraph of a graph:

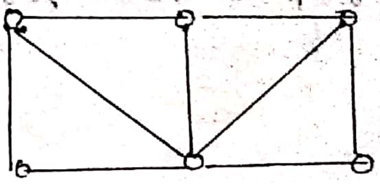
Let $G = (V, E, \gamma)$ be a graph. A triple (V_1, E_1, γ_1) is called a subgraph of G if V_1 is a non-empty subset of V, E_1 is a subset of E, γ_1 is a restriction of γ to E_1 such that for all $e \in E_1$ if $\gamma_1(e) = \{u, v\}$ then $u, v \in V_1$



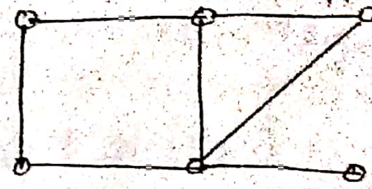
G G1 G2 G3

The graphs G_1, G_2 & G_3 are the subgraphs of the graph G

A spanning subgraph of a graph G is a graph containing all pts of G.



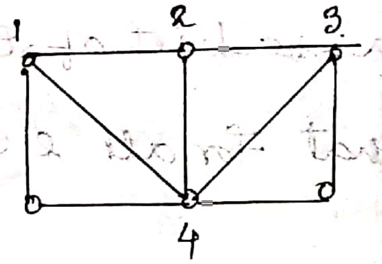
G (V, X)



G' (V, X'), X' ⊂ X

spanning subgraph of G.

For any set S of V, S ⊂ V, the induced subgraph ⟨S⟩, is the maximal subgraph of G with point set S. Two points of ⟨S⟩ are adjacent iff they are adjacent in G.



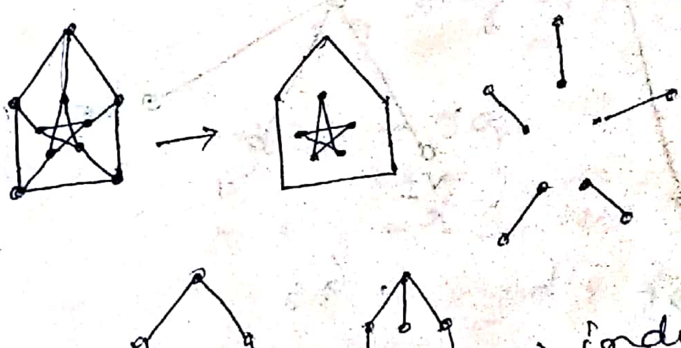
G (V, X)

⟨S⟩

V = S = {1, 2, 3, 4}

induced subgraph of G

G :



[spanning subgraphs of G]

→ induced subgraph of G.

Isomorphic Graph

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. A function $f = V_1 \rightarrow V_2$ is called a graphs isomorphism if (*a*) f is one-to-one onto and (*b*) for all $a, b \in V_1$ $\{a,b\} \in E_1$ if and only if $\{f(a), f(b)\} \in E_2$. When such a function exists, G_1 and G_2 are called isomorphic graphs and is written as $G \cong G_2$

Example 18. Show that the two graphs shown in figure are isomorphic

Here, $V(G_1) = \{1,2,3,4\}$, $V(G_2) = \{a,b,c,d\}$, $E(G_1) = \{\{1,2\}, \{2,3\}, \{3,4\}\}$ and $E(G_2) = \{\{a,b\}, \{b,d\}, \{d,c\}\}$

Define a function f: $V(G_1) \rightarrow V(G_2)$ as

$f(1) = a, f(2) = b, f(3) = d$ and $f(4) = c$.

f is clearly one–one and onto, hence an isomorphism.

Further,

$\{1,2\} \in E(G_1)$ and $\{f(1), f(2)\} = \{a, b\} \in E(G_2)$

$\{2,3\} \in E(G_1)$ and $\{f(2), f(3)\} = \{b, d\} \in E(G_2)$

$\{3,4\} \in E(G_1)$ and $\{f(3), f(4)\} = \{d, c\} \in E(G_2)$

and

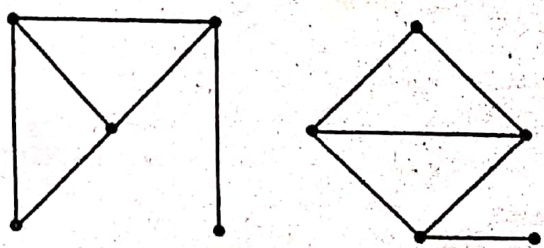
$\{1,3\} \notin E(G_1)$ and $\{f(1), f(3)\} = \{a, d\} \notin E(G_2)$

$\{1,4\} \notin E(G_1)$ and $\{f(1), f(4)\} = \{a, c\} \notin E(G_2)$

$\{2,4\} \notin E(G_1)$ and $\{f(2), f(4)\} = \{b, c\} \notin E(G_2)$.

Hence f preserves adjacency as well as non-adjacency of the vertices. Therefore G_1 and G_2 are isomorphic.

We hereby give some examples of isomorphic graphs.



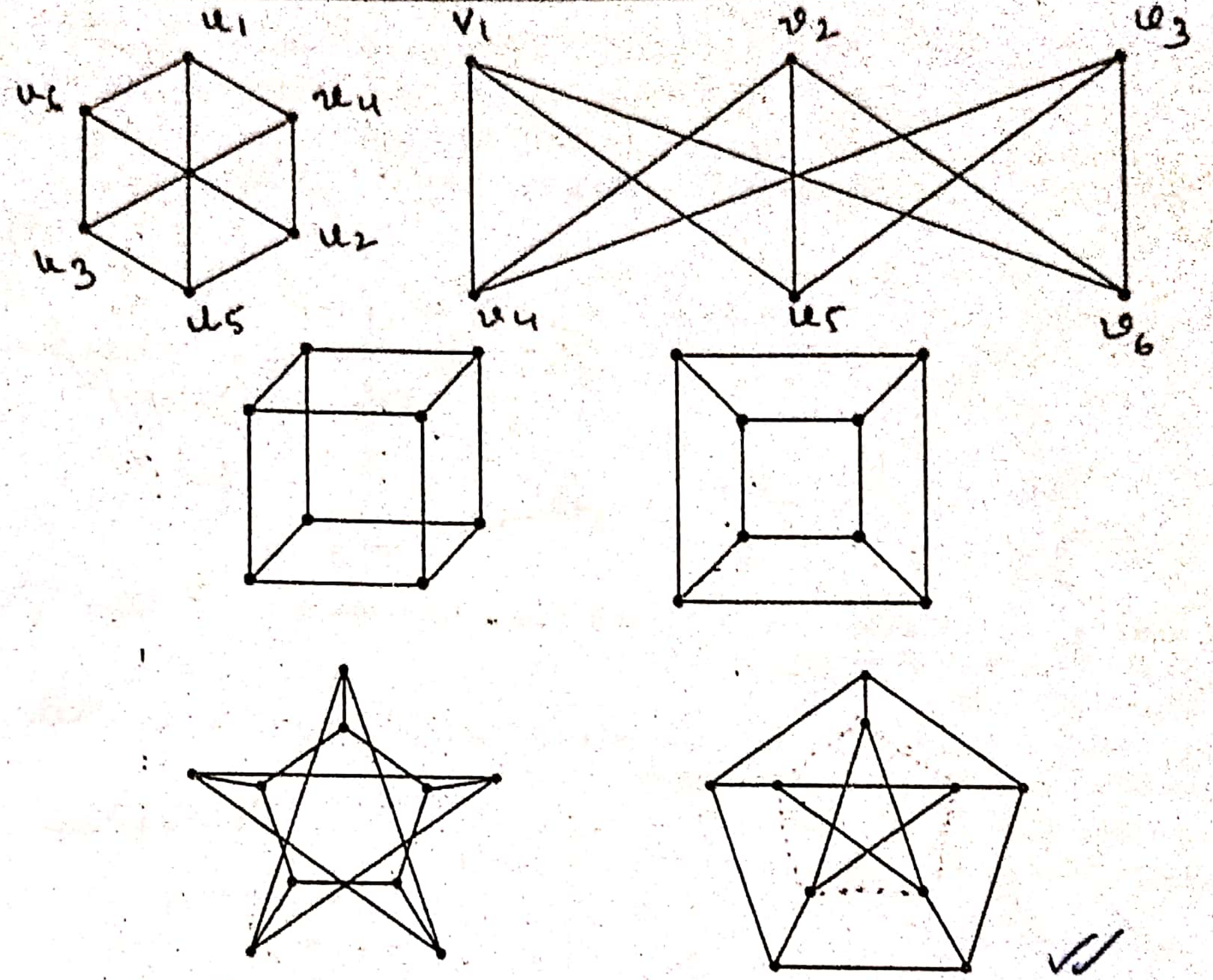


Fig. 13.23. Isomorphic pairs of graphs

It is not always an easy task to determine whether or not two given graphs are isomorphic. However, we can prove that two graphs are not isomorphic by showing that they do not share a property that isomorphic graphs must both have, such a property is called an **invariant.** Since an ismorphism is a one to one onto function between vertex sets, isomorphic graphs have the same numbers of vertices. Many other properties are shared by isomorphic graphs.

Suppose ϕ: $G_1 \to G_2$ is an isomorphism from a graph G_1 to G_2. If v is a vertex of degree k in G_1 and if $v_1, v_2, \ldots, v_k$ are the vertices adjacent to v, then, in G_2, $\phi(v)$ is adjacent to the k vertices $\phi(v_1)$, $\phi(v_2), \ldots \phi(v_k)$, but to no other vertex. Thus, the degree of $\phi(v)$ is also k. It follows that isomorphic graphs have the same degree sequences and hence also the same numbers of edges, since the number of edges in a graph is one half the sum of the vertex degrees.

If G_1 and G_2 are isomorphic graphs, then G_1 and G_2 have the

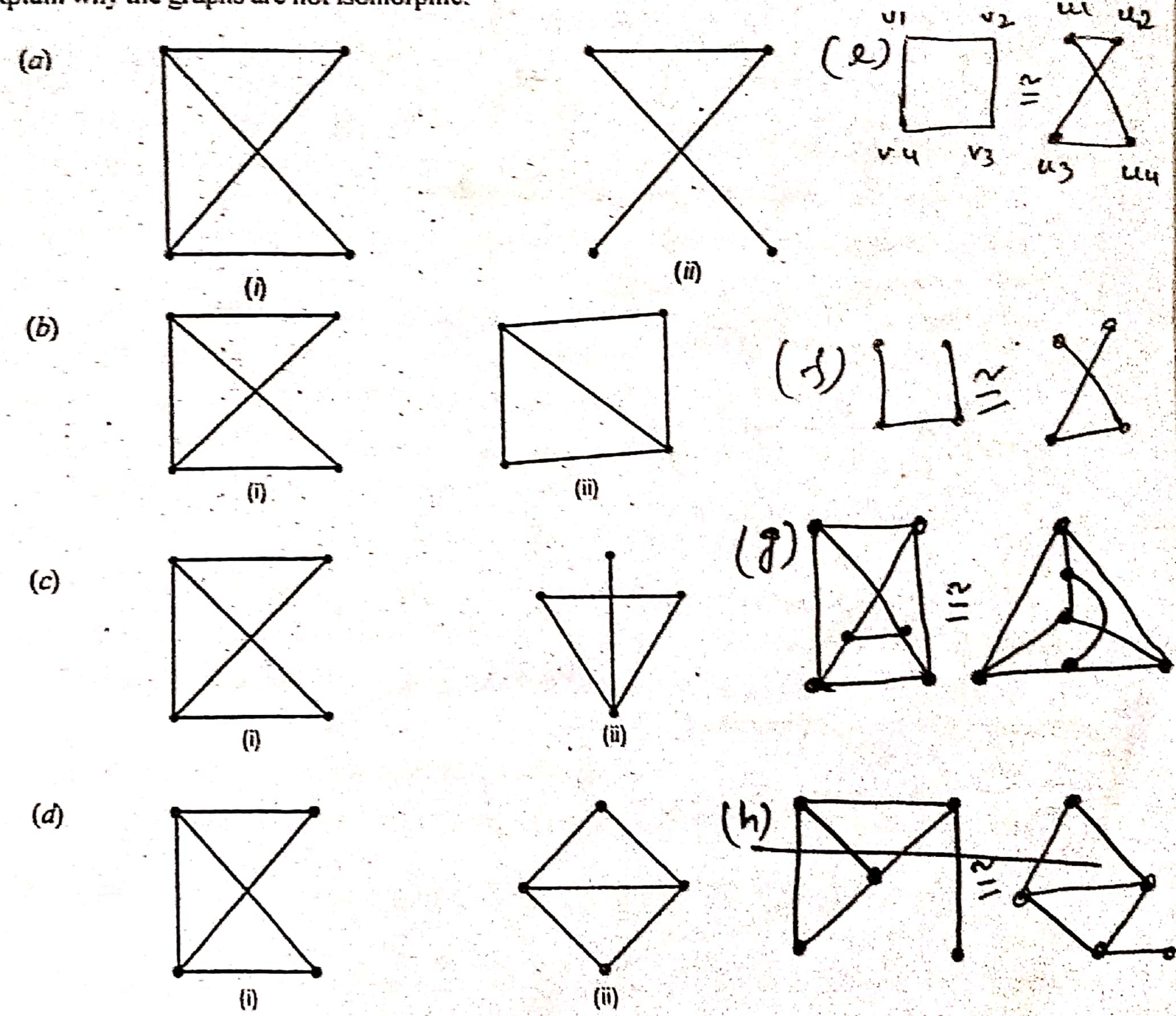
- same number of vertices,
- same number of edges, and
- same degree sequences.

If any of these quantities differ in two graphs, they can not be isomorphic. However these conditions are by no means sufficient. For instance, the two graphs shown in Fig. 13.24 satisfy all three conditions., yet they are not isomorphic. That the graphs in Fig. 13.24 (a) and (b) are not isomorphic can be shown as follows : if the graph in Fig. 13.24 (a) were to be isomorphic to the one in (b), vertex x must correspond to y, because there are no other vertices of degree three. Now in (b) there is only one pendant vertex, w, adjacent to y, while in (a) there are two pendant vertices, u and v, adjacent to x.

Fig. 13.24. Two graphs that are not isomorphic

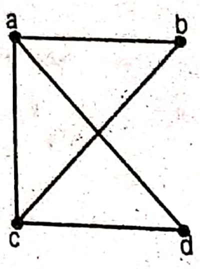
Although every known algorithms to test whether two graphs are isomorphic requires exponential or factorial time in the worst case, there are algorithms that can determine whether a pair of graphs is isomorphic in linear time in the average case. Finding a simple and efficient criteria for detection of isomorphism is an important unsolved problem in graph theory.

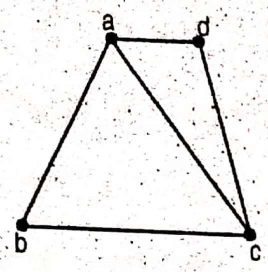
Example 19. For each pair of graphs shown, either label the graphs so as to exibit an isoporphism or explain why the graphs are not isomorphic.



Solution : (*a*) The graphs are not isomorphic because (*i*) has five edges and (*ii*) has four edges.

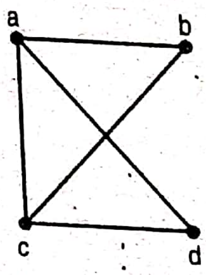
(*b*) the graphs are isomorphic, as shown by the labelling

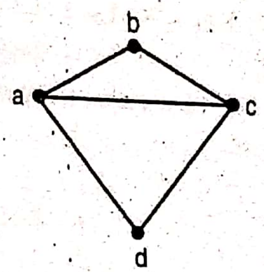




(*c*) The graphs are not isomorphic because (*ii*) has a vertex of degree 1 and (*i*) does not have.

(*d*) The graphs are isomorphic, as shown by the labellling.





13.7. Operations of Graphs