Density of distribution in phase space :-

The use of ensembles in st. mechanics is guided by the following factors →

① There is no need to maintain distinction between individual systems as we are only interested in number of system at any time which would be found in different state that corresponding different phase space.

② The no. of elements in an ensemble is so large that there is a continuous change in this number in passing from one region of phase to another.

The condition of an ensemble at any time can be specified by the density ρ with the phase points are distributed over the phase space named as Density of Distribution or Probability Density of Distribution Function.

The density of distribution ρ is the function of f position coordinates and f-momentum co-ordinate i.e $(q_1, q_2 \ldots q_f)$ and $(p_1, p_2 \ldots p_f)$ corresponding to 2f axes in phase space.

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The density of distribution is also a function of time because at any time i.e

$$\rho = \rho\{(q_1 \cdots q_f) : p_1, p_2 \cdots p_f) : t\}$$

$$\Rightarrow \rho = \rho(q, p, t)$$

The small volume of the phase space named as Hyper-volume $\delta\tau$ of the phase points -

$$\delta\tau = \delta q_1 \, \delta q_2 \cdots \delta q_f ; \delta p_1 \cdots \delta p, t$$

The number of systems δN lying in the specified region can be obtained by multiplying the density of distribution and the Hyper-volume in the phase space i.e

$$\delta N = \rho(q, p, t) \, \delta q_1 \cdots \delta q_f , \delta p_1 \cdots \delta p_f$$

$$\therefore \delta N = \rho \, \delta q_1 \cdots \delta q_f \, \delta p_1 \cdots \delta p_f$$

In brief

$$\delta N = \rho \prod_{i=1}^{f} \delta q_i \, \delta p_i$$

By integrating over the whole of the phase space

$$N = \int \rho \, dq_1 \cdots dq_f \, dp_1 \cdots dp_f$$

** General discussion of Mean Values :-

The average value of variable u in an ensemble is defined by multiplying each possible u_r by the number n_r of the system in the ensemble which exhibits this value, adding the resultant product for all possible values of the variable u and then dividing this sum by the total number of system in the ensemble.

Let u be the variable which can assume any of the M discrete values.

$u_1, u_2 \,--\, u_M$

with respective probabilities

$w(u_1) \cdots w(u_M)$

Then mean (or average) value of u is denoted by $\bar{u}$ and is defined by

$$\bar{u} = \frac{w(u_1)u_1 + w(u_2)u_2 \cdots\cdots w(u_M)u_M}{w(u_1) + w(u_2) + \cdots + w(u_M)}$$

$$\bar{u} = \frac{\sum_{i=1}^{M} w(u_i)\, u_i}{\sum_{i=1}^{M} w(u_i)}$$

More generally, if $f(u)$ is any function of u [th]en the mean value $f(u)$ is denoted by

$$\overline{f(u)} = \frac{\sum_{i=1}^{M} w(u_i) f(u_i)}{\sum_{i=1}^{M} w(u_i)}$$

This expression can be simplified as unity,

$$\sum_{i=1}^{M} w(u_i) = w(u_1) + w(u_2) + \cdots + w(u_M) = 1$$

$$\therefore \quad \overline{f(u)} = \sum_{i=1}^{M} w(u_i) f(u_i)$$

If the Probability distribution function is Continuous in all position and momentum co-ordinates the above eqn →

$$\bar{u} = \frac{\int u(q,p)\, w(q,p)\, d\Gamma}{\int w(q,p)\, d\Gamma}$$

where $d\Gamma = dq_1 \cdots dq_M, dp_1 \cdots dp_M$.

But according to normalisation Condition

$$\int w(q,p)\, d\Gamma = 1$$

$$\bar{u} = \int u(q,p)\, w(q,p)\, d\Gamma$$

If $f(u)$ and $g(u)$ are any two function of u, then

$$\overline{f(u)+g(u)} = \sum_{i=1}^{M} w(u_i)\left[f(u_i) + g(u_i)\right]$$

$$= \sum_{i=1}^{M} w(u_i) f(u_i) + \sum_{i=1}^{M} w(u_i) g(u_i)$$

Suppose we have to determine the average distance from the origin with total points N

$$\bar{x} = \frac{\sum_{i=1}^{N} x(i)}{N}$$

Now, if the line is divided into cell, and the distribution is given in terms of the number of points in each cell, then

$$\bar{x} = \frac{\sum_{i} N_i\, x(i)}{N}$$

$$\Rightarrow \quad \bar{x} = \frac{\int_{-\infty}^{\infty} x\, n(x)\, dx}{N}$$

, Noting that

$$\int_{-\infty}^{\infty} n(x)\, dx = N$$

$$\bar{x} = \frac{\int_{-\infty}^{\infty} x\, n(x)\, dx}{\int_{-\infty}^{\infty} n(x)\, dx}$$

Macroscopic behaviour as an average over microscopic behaviour :- In macroscopic point of view the thermodynamic properties remain constant but in microscopic point of view ~~next~~ never stay constant. One can only define the probability for the set of all possible microscopic states of the systems.

The microscopic value of x is represented by $x(q, p) = x(P)$ where P is the phase point. The observed value x_{obs} in macroscopic sense should be the average value of microscopic x i.e.

$$x_{obs} = \bar{x}$$

Let M is the set of all microscopic states which can be realised by the system under a given macroscopic conditions and probability dw over the volume element $d\Gamma$, where $d\Gamma = dq_1 \ldots dq_f \, dp_1 \ldots dp_f$

$$dw = \int_{\Delta\Gamma} \rho(P) \, d\Gamma,$$

$\Delta\Gamma$ belongs to M and ρ is probability density and whole phase

$$\int \rho \, d\Gamma = 1$$

$$\therefore \bar{x} = \int_M x(P) \, \rho(P) \, d\Gamma$$

LIOUVILLES THEOREM

In dynamical state of a system the point in phase space is not stationary but it will move along a definite trajectory path which is determined from the equation of motion,

$$\dot{q}_i = \frac{\partial H}{\partial P_i} \quad \& \quad \dot{P}_i = \frac{\partial H}{\partial q_i}$$

where $H = H(q_1 \ldots q_f ; P_1 \ldots P_f)$ is the Hamiltonian of the system.

As a result of this motion the density ρ of system in phase space changes with time. In finding $\frac{\partial \rho}{\partial t}$ at a given point in phase Liouville's made Theorem — em

This theorem is primarily concerned with defining a fundamental property of the phase space — ie space for position & ~~mo~~ space for momentum coordinate and hence consists of two part

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First part :- The first part states the conservation of density in phase space ie

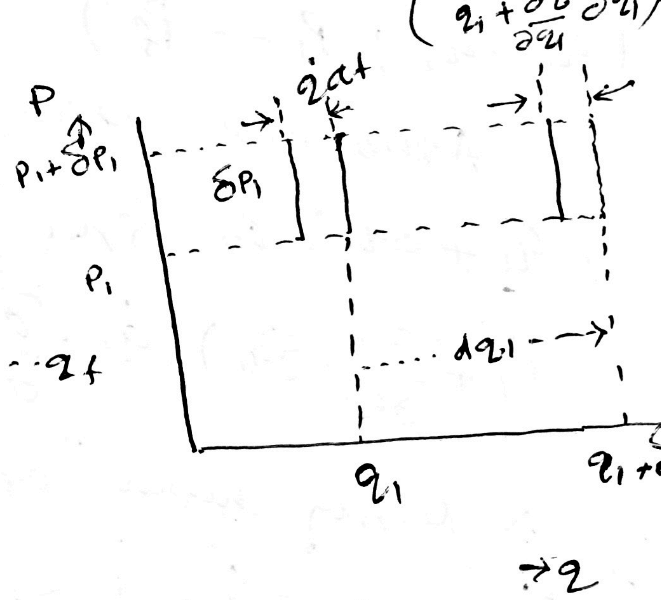
$$\frac{d\rho}{dt} = 0.$$

2nd Part :- The second part gives the conservation of extension in phase space ie

$$\frac{d(\delta \rho)}{dt} = 0$$

or the volume at the disposal of a particular number of phase points is conserved through the phase space

Proof of 1st Part :-



Consider any fixed element of volume of phase space located betn q_1 & $q_1 + \delta q_1 \cdots q_f$ and p_1 & $p_1 + \delta p_1 \cdots\cdots p_f$

The no. of volume

$$(\delta q_1 \cdots \delta q_f ; \delta p_1 \cdots \delta p_f)$$

The change in number of systems within volume of phase space is given by

$$\left(\frac{\delta \rho}{\partial t}\right) dt \left(\delta p_1 \cdots \delta p_f\right)$$

This change is due to the number of system entering and leaving this volume in time dt. Let two faces of hyper volume normal to the q_1 axis with q_1 and $q_1 + \delta q_1$. Hence the no. of phase points entering the first face in time will be

$$\rho \dot{q}_1 dt (\delta q_2 \cdots \delta q_f)(\delta p_1 \cdots \delta p_f)$$

where ρ and $\dot{q}_1$ are the density and indicated component of velocity for representive points ~~$(q_1 \cdots q_f)$~~ $(q_1 \cdots q_f ; p_1 \cdots p_f)$

Again the phase points leaving the face $(q_1 + \delta q_1)$ in time dt will be

$$\left(\rho + \frac{\partial \rho}{\partial q_1}\delta q_1\right)\left(\dot{q}_1 + \frac{\partial \dot{q}_1}{\partial q_1}\delta q_1\right) dt\, \delta q_2 \cdots \delta q_f\, \delta p_1 \cdots \delta p_f \quad -(1)$$

Neglecting higher order terms, we have

$$\left[\rho \dot{q}_1 + \left(\rho \frac{\partial \dot{q}_1}{\partial q_1} + \dot{q}_1 \frac{\partial \rho}{\partial q_1}\right)\delta q_1\right] dt\, \partial q_2 \cdots \delta q_f\, \delta p_1 \cdots dp_f \quad -(2)$$

Substracting (2) from eqn (1), we have

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$$-\left(\rho \frac{\partial \dot{q}_1}{\partial q_1} + \dot{q}_1 \frac{\partial \rho}{\partial q_1}\right) dt\, \delta q_1 \ldots \delta q_f\, \delta p_1 \ldots \delta p_f$$

Illy for p_1 co-ordinate, we have

$$-\left(\rho \frac{\partial \dot{p}_1}{\partial p_1} + \dot{p}_1 \frac{\partial \rho}{\partial p_1}\right) dt\, \delta q_1 \ldots \delta q_f\, \delta p_1 \ldots \delta p_f$$

The net increase in time dt of number of system in this volume of phase space is then obtained by summing the net number of system entering the volume through all the faces labelled by $q_1 \ldots q_f$ and $p_1 \ldots p_f$. Then

$$\frac{d}{dt}(\delta N) = -\sum_{i=1}^{f}\left\{\rho\left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i}\right) + \left(\frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i\right)\right\} dt\, \delta q_1 \ldots \delta q_f\, \delta p_1 \ldots \delta p_f \quad \text{— (3)}$$

Now $\frac{d}{dt}(\delta N) = \frac{\partial \rho}{\partial t} dt\, \delta q_1 \ldots \delta q_f\, \delta p_1 \ldots \delta p_f$

$$\frac{\partial \rho}{\partial t} dt\, \delta q_1 \ldots \delta q_f\, \delta p_1 \ldots \delta p_f = -\sum_{i=1}^{f}\left\{\rho\left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i}\right) + \left(\frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i\right)\right\} dt\, \delta q_1 \ldots \delta q_f\, \delta p_f \ldots \delta p_f \quad \text{—}$$

$$\frac{\partial \rho}{\partial t} = -\sum_{i=1}^{f}\left\{\rho\left(\frac{\partial \dot{p}_i}{\partial p_i} + \frac{\partial \dot{q}_i}{\partial q_i}\right) + \left(\frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i\right)\right\} \quad \text{— (3)}$$

The equations of motion in canonical form

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\frac{\partial \dot{q}_i}{\partial q_i} = \frac{\partial^2 H}{\partial q_i \partial p_i}$$

Since the order of differential is immaterial

$$\therefore \sum_{i=1}^{f} \left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right) = 0 \text{ ——— } (4)$$

Substituting eqn (4) in eqn (3)

$$\left(\frac{\partial \rho}{\partial t} \right)_{q,p} = -\sum_{i=1}^{f} \left(\frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \right) \text{ —— } (5)$$

This result is known as Liouville's theorem. Hence eqn (5) can be written as

$$\left(\frac{\partial \rho}{\partial t} \right)_{q,p} + \sum_i \frac{\partial \rho}{\partial q_i} \frac{\partial q_i}{\partial t} + \sum \frac{\partial \rho}{\partial p_i} \frac{\partial p_i}{\partial t} = 0 \text{ —— } (6)$$

and is identical with the equation of continuity in hydrodynamics. If ρ is a function of q, p & t and q & p are the function of 't' then

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial q} \frac{dq}{dt} + \frac{\partial \rho}{\partial p} \frac{dp}{dt}$$

generalising this for all f's in q's we get

$$\Rightarrow \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \sum_{i=1}^{f} \frac{\partial \rho}{\partial q_i} \frac{dq_i}{dt} + \sum_{i=1}^{f} \frac{\partial \rho}{\partial p_i} \frac{dp_i}{dt} \text{ ——— } (7)$$

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comparing the eqn (6) & (7) we have that

$$\frac{d\rho}{dt} = 0$$

This form of expression may be called the principle of the conservation of density in phase space. Therefore, the density of a group of points remains constant along their trajectory in the phase space. ie ~~will~~ ever have uniform density.

=== x ===

__2nd part :-__ Since we have

$$\delta N = \rho\, \delta\Gamma$$

$$\Rightarrow \frac{d}{dt}(\delta N) = \frac{d\rho}{dt}\delta\Gamma + \rho \frac{d}{dt}(\delta\Gamma)$$

Since the number of phase points δN in a given region of the phase space must remain constant. fixed, as the system can neither be created nor destroyed, we have

have $$\frac{d}{dt}(d\Gamma) = 0$$

$$\Rightarrow \frac{d\rho}{dt}(d\Gamma) + \rho \frac{d}{dt}(\delta\Gamma) = 0$$

We have proved that $\frac{d\rho}{dt} = 0$, Hence it follows

that, $\rho \neq 0$

$$\therefore \frac{d}{dt}(\delta\Gamma) = 0$$

This equation gives the principle of conservation of extension in phase space. //

Postulate of equal A Priori Probability :—

So far the methods used in st. mechanics are chiefly based on the Principal of classical mechanics. In order to make applications to situations of actual interest we must now understand a consideration of the fundamental hypothesis as to equal a "PRIORI PROBABILITIES" for equal regions in phase space. This postulate doesn't arise due to any inadequacy in the principles of classical mechanics but due to the incompleteness of our knowlege concerning the systems.

A/c to this postulate the probability of finding the phase point for given systems in any region of the phase space is identical with that for any other region of equal extension or volume.

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