Interpolation with Unequal Intervals

Defination: Divided Difference:

Let $f(x_0), f(x_1), \ldots, f(x_n)$ be the values of the function $f(x)$ corresponding to the values $x_0, x_1, x_2, \ldots, x_n$ of x, where $x_1 - x_0$, $x_2 - x_1$, $x_3 - x_2, \ldots, x_n - x_{n-1}$ are not necessarily equal.

Then first divided difference (d.d.) of $f(x)$ for the arguments x_0 and x_1 denoted by $\underset{x_1}{\Delta} f(x_0)$ or $f(x_0, x_1)$ is defined by

$$f(x_0, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

The 2^{nd} devided difference (2nd d.d.) of $f(x)$ for the arguments x_0, x_1 and x_2 denoted by $\underset{x_1 x_2}{\Delta^2} f(x_0)$ or $f(x_0, x_1, x_2)$ is defined as

$$f(x_0, x_1, x_2) = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_2}$$

Similarly, n'th devided difference for the arguments $x_0, x_1, x_2, \ldots, x_n$ is defined as

$$f(x_0, x_1, \ldots, x_n) = \frac{f(x_0, x_1, \ldots, x_{n-1}) - f(x_1, x_2, \ldots, x_n)}{x_0 - x_n}$$

Formation of divided difference table :-

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
x_0	$f(x_0)$				
		$f(x_0, x_1)$			
x_1	$f(x_1)$		$f(x_0, x_1, x_2)$		
		$f(x_1, x_2)$		$f(x_0, x_1, x_2, x_3)$	
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$		$f(x_0, \dots, x_4)$
		$f(x_2, x_3)$		$f(x_1, x_2, x_3, x_4)$	
x_3	$f(x_3)$		$f(x_2, x_3, x_4)$		
		$f(x_3, x_4)$			
x_4	$f(x_4)$				

Ex. Form a d.d. for the following data:

x:	2	4	9	10
$f(x)$:	4	56	711	980

Sol^n.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
2	4			
		$\frac{56-4}{4-2} = 26$		
4	56		$\frac{131-26}{9-2} = 15$	
		$\frac{711-56}{9-4} = 131$		$\frac{23-15}{10-2} = 1$
9	711		$\frac{269-131}{10-4} = 23$	
		$\frac{980-711}{10-9} = 269$		
10	980			

Ex. Find the first three d.d. for the function $\frac{1}{x^2}$ for the arguments a, b, c, d.

Sol: Let $f(x) = \frac{1}{x^2}$.

$$\therefore \text{1st. d.d.} = f(a,b) = \frac{\frac{1}{b^2} - \frac{1}{a^2}}{b-a}$$

$$= \frac{a^2-b^2}{b^2a^2(b-a)} = -\frac{a+b}{a^2b^2}, \quad a \neq b.$$

$$f(a,b,c) = \frac{f(b,c) - f(a,b)}{c-a}$$

$$= \frac{-\frac{b+c}{b^2c^2} + \frac{a+b}{a^2b^2}}{c-a} = \frac{-a^2(b+c) + c^2(a+b)}{a^2b^2c^2(c-a)}$$

$$= \frac{ac(c-a) + b(c^2-a^2)}{a^2b^2c^2(c-a)} = \frac{(c-a)(ab+bc+ca)}{a^2b^2c^2(c-a)}$$

$$= \frac{ab+bc+ca}{a^2b^2c^2}$$

$$f(a,b,c,d) = \frac{f(b,c,d) - f(a,b,c)}{d-a}$$

$$= \frac{\frac{bc+cd+db}{b^2c^2d^2} - \frac{ab+bc+ca}{a^2b^2c^2}}{d-a}$$

$$= \frac{a^2bc + a^2cd + a^2bd - d^2ab - d^2bc - d^2ca}{a^2b^2c^2d^2(d-a)}$$

$$= \frac{-dab(d-a) - bc(d^2-a^2) - dca(d-a)}{a^2b^2c^2d^2(d-a)}$$

$$= \frac{(d-a)[-dab - bcd - bca - dca]}{a^2b^2c^2d^2(d-a)}$$

$$= -\frac{abc+bcd+cda+dab}{a^2b^2c^2d^2}$$

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