Fermat's Last theorem for $n=4$

Firstly we will establish the fact that it is impossible to solve the eqn $x^4+y^4=z^2$ in the +ve integers.

Theorem (Fermat) The Diophantine eqn $x^4+y^4=z^2$ has no soln in positive integers x, y, z.

Pf The given Diophantine eq. is

$$x^4+y^4=z^2 \quad -(1)$$

Suppose (1) has a +ve non-trivial soln say x_0, y_0, z_0

Let $\gcd(x_0, y_0)=1$

We have $\quad x_0^4+y_0^4=z_0^2$

$$\Rightarrow (x_0^2)^2+(y_0^2)^2=z_0^2$$

Clearly x_0^2, y_0^2, z_0 is a Pythagorean triple and $\gcd(x_0^2, y_0^2, z_0)=$

[If $\gcd(x_0^2, y_0^2, z_0)\neq 1$ then let $\gcd(x_0^2, y_0^2, z_0)=d>1$
$\therefore \exists$ prime p s.t. $p|d$. Hence $p|x_0^2$, $p|y_0^2$, $p|z_0$
$\therefore$ $\gcd(x_0^2, y_0^2)\geq p>1 \Rightarrow \gcd(x_0, y_0)>1$ which is a contradiction]

Hence x_0^2, y_0^2, z_0 is a primitive Pythagorean triple.

$\therefore$ One of the integers x_0^2 or y_0^2 is even and the other is odd.

Suppose x_0^2 is even. Then ~~the~~ we have

$$x_0^2=2st \quad y_0^2=s^2-t^2 \quad \text{and} \quad z_0=s^2+t^2$$

for some $s>t>0$ $s, t\in \mathbb{Z}$

and $\gcd(s, t)=1$

and $s\not\equiv t \pmod 2$

[i.e. one of s and t is even]

$\because$ y_0^2 is odd (as we have assumed that x_0^2 is even)

$\therefore$ $y_0^2 \equiv 1 \pmod 4$

$\Rightarrow 1 \equiv y_0^2 \pmod 4$

$\Rightarrow 1 \equiv s^2 - t^2 \pmod 4$

Now ~~since~~ let us take s to be even and hence t to be odd

$\therefore$ We can write

$$s^2 \equiv 0 \pmod 4 \quad \text{and} \quad t^2 \equiv 1 \pmod 4$$

$\therefore$ $1 \equiv 0 - 1 \equiv -1 \pmod 4$

$\Rightarrow 1 \equiv 3 \pmod 4$

which is impossible.

$\therefore$ s^2 must be odd (hence s must be odd)

and t^2 " " even ($\because$ t " " even)

Let us put $t = 2r$ $(r \in \mathbb{Z})$. This gives

$$x_0^2 = 2st = 4sr$$

$$\Rightarrow \left(\frac{x_0}{2}\right)^2 = sr$$

[Note that $\gcd(s, r) = 1$]

Hence $s = z_1^2$ and $r = w_1^2$ for some $z_1, w_1 \in \mathbb{N}$

Now we observe the eq^n

$$t^2 + y_0^2 = s^2 \qquad -(2)$$

Here $\gcd(s, t) = 1 \Rightarrow \gcd(t, y_0, s) = 1$

Hence t, y_0, s is a primitive Pythagorean triple and

$t = 2uv$ $\quad$ $y_0 = u^2 - v^2$ $\quad$ $s = u^2 + v^2$ for some $u, v \in \mathbb{Z}$ s/

$u > v > 0$

$\gcd(u, v) = 1$ and

$u \not\equiv v \pmod 2$

Note that $uv = \frac{t}{2} = r = w_1^2$

i.e. $uv = w_1^2$

Hence $\exists$ x_1 and y_1 s.t. $(x_1, y_1 \in \mathbb{N})$

$u = x_1^2$ and $v = y_1^2$

$\therefore$ $z_1^2 = s = u^2 + v^2 = x_1^4 + y_1^4$

$\because$ z_1 and t are positive we have (Clearly $z_1 \geq 1$, $t \geq 1$)

$$0 < z_1 \leq z_1^2 = s \leq s^2 < s^2 + t^2 = z_0$$

Hence now we have another sol of eq^n ① viz x_1, y_1, z_1 s.t. $z_0 > z_1 > 0$. Repeating the whole argument will give us another sol of eq^n ① say x_2, y_2, z_2 s.t. $z_0 > z_1 > z_2 > 0$. Repeating this process gives us infinite decreasing seq. of +ve numbers s.t.

$$z_0 > z_1 > z_2 > \cdots > 0$$

But there are only finite positive integers less than z_0. Hence we have a contradiction. Therefore eq^n ① has no sol^n in positive integers.

Cor: The eq^n $x^4 + y^4 = z^4$ has no sol^n in +ve integers

Pf/ Suppose x_0, y_0, z_0 is a positive sol^n of $x^4 + y^4 = z^4$

Then x_0, y_0, z_0^2 is a positive sol. of $x^4 + y^4 = z^2$ which is impossible

Therefore the given eq^n has no sol^n in +ve integers.