* Note. If x, y, z is a primitive Pythagorean triple then exactly one of the integers x or y is divisible by 3.

Pf. From the previous thm we have

$$x = 2st \quad y = s^2 - t^2 \text{ and } z = s^2 + t^2$$

where $s > t > 0$ $\gcd(s,t) = 1$ and $s \not\equiv t \pmod 2$

Case I Either $3/s$ or $3/t$

Then $3/x$

Case II Neither $3/s$ nor $3/t$

Fermat's little thm asserts that

$$s^{3-1} \equiv 1 \pmod 3 \qquad t^{3-1} \equiv 1 \pmod 3$$

i.e. $s^2 \equiv 1 \pmod 3$ $\qquad t^2 \equiv 1 \pmod 3$

$\therefore$ $y = s^2 - t^2 \equiv 1 - 1 = 0 \pmod 3$

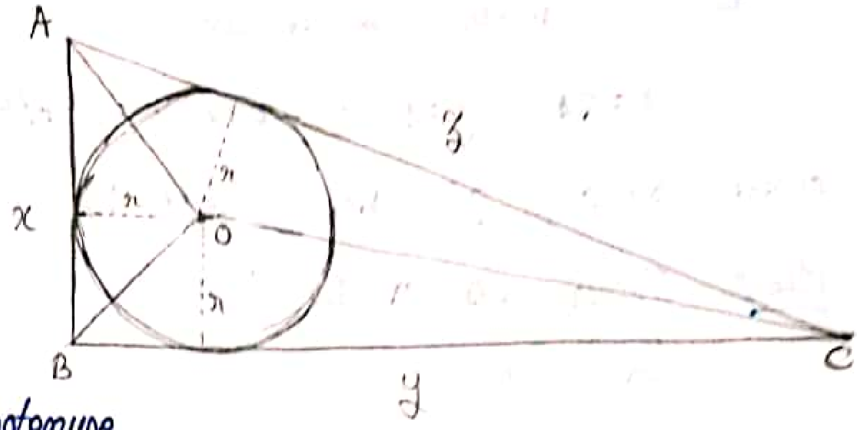
i.e. $y \equiv 0 \pmod 3$

i.e. $3 \mid y - 0 = y$

Defn (Pythagorean triangle) It is a right triangle whose sides are of integral length

Thm The radius of the inscribed circle of a Pythagorean triangle is always an integer.

Pf Let r be the radius of the inscribed circle



The triangle in the given figure has z as length of the hypotenuse and x, y as length of the other two sides

From the figure we can see that

$$ar(\Delta ABC) = ar(\Delta AOB) + ar(\Delta BOC) + ar(\Delta COA)$$

$$\Rightarrow \frac{1}{2}xy = \frac{1}{2}rx + \frac{1}{2}ry + \frac{1}{2}rz = \frac{1}{2}r(x+y+z) \quad -(1)$$

We know that $x^2+y^2=z^2$

and +ve integral sol^n of this eq^n are given by

$$x = 2kst \quad y = k(s^2-t^2) \quad z = k(s^2+t^2)$$

where k, s, t are +ve. Putting these values of x, y, z in (1):

$$\frac{1}{2}(2kst)k(s^2-t^2) = \frac{1}{2}r\left(2kst + k(s^2-t^2) + k(s^2+t^2)\right)$$

$$\Rightarrow k^2st(s^2-t^2) = \frac{1}{2}r(2kst + 2ks^2)$$

$$\Rightarrow \cancel{k^2}^{k}\cancel{s}t(s^2-t^2) = \cancel{\frac{1}{2}}r \cdot \cancel{2k}\cancel{s}(t+s)$$

$$\Rightarrow kt(s^2-t^2) = r(t+s)$$

$$\Rightarrow kt(s-t) = r$$

Hence r is an integer