Some properties of the phi-function

Thm (Gauss) For each +ve integer $n \geq 1$

$$n = \sum_{d|n} \phi(d)$$

the sum being extended over all +ve divisors of n

Pf **Case I** $n = 1$ $\quad \sum_{d|n} \phi(d) = \sum_{d|1} \phi(d) = \phi(1) = 1 = n$

Case II $n > 1$ Consider the f^n

$$F(n) = \sum_{d|n} \phi(d)$$

$\because$ ϕ is multiplicative therefore F is also multiplicative

Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ be the prime factorization of n. Then

$$F(n) = F(p_1^{k_1}) F(p_2^{k_2}) \cdots F(p_r^{k_r})$$

For each i

$$F(p_i^{k_i}) = \sum_{d|p_i^{k_i}} \phi(d)$$

$$= \phi(1) + \phi(p_i) + \phi(p_i^2) + \cdots + \phi(p_i^{k_i})$$

$$= \cancel{1} + \cancel{p_i} - \cancel{1} + p_i^2 - \cancel{p_i} + \cdots + p_i^{k_i} - p_i^{k_i - 1}$$

$$= p_i^{k_i}$$

$\therefore$ $F(n) = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} = n$.

i.e. $\quad n = \sum_{d|n} \phi(d)$

Thm For $m>1$ the sum of the +ve integers less than m and relatively prime to m is $\frac{1}{2}m\phi(m)$.

Pf Let $a_1, a_2, \ldots, a_{\phi(m)}$ be the positive integers less than m and relatively prime to m.

We know that $\gcd(a, m)=1$ iff $\gcd(m-a, m)=1$ [Use division algorithm to find gcd technique]

[Note that since $a_i<m$ therefore $m-a_i<m$]

Hence $m-a_i \equiv a_j \pmod{m}$ $\quad 1\leq i,j\leq \phi(m)$

i.e. $m-a_i = a_j$ $\quad$ ($\because$ $m-a_i, a_j$ are both $<m$)

$\forall\ i,j$

i.e. the numbers $m-a_1, m-a_2, \ldots, m-a_{\phi(m)}$ are equal in some order to $a_1, a_2, \ldots, a_{\phi(m)}$. Thus

$$a_1+a_2+\cdots+a_{\phi(m)} = (m-a_1)+(m-a_2)+\cdots+(m-a_{\phi(m)})$$

$$= \phi(m)\cdot m - (a_1+a_2+\cdots+a_{\phi(m)})$$

$$\Rightarrow 2(a_1+a_2+\cdots+a_{\phi(m)}) = \phi(m)\cdot m$$

Example $m=30$. $\phi(30)=\phi(5\times 6) = \cancel{(5-1)}$

$$=\phi(5\times 3\times 2) = (5-1)(3-1)(2-1) = 4\cdot 2\cdot 1 = 8$$

There are 8 +ve integers less than 30 and relatively prime to 30 which are:

1, 7, 11, 13, 17, 19, 23, 29

Hence $1+7+11+13+17+19+23+29 = 120 = \frac{1}{2}\cdot 30\cdot 8$

Also note $\quad 1+29=30 \quad 11+19=30$

$\quad 7+23=30 \quad 13+17=30$

Scanned by CamScanner

Thm For any +ve integer n

$$\phi(n) = n \sum_{d|n} \frac{\mu(d)}{d}$$

Pf. We have $F(n) = n = \sum_{d|n} \phi(d)$

Applying the inversion formula

$$\phi(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d) \frac{n}{d}$$