D) The Greatest Integer function

For $x \in \mathbb{R}$ $[x]$ denotes the largest integer less than or equal to x

i.e. x is the unique integer satisfying $x-1 < [x] \le x$

It is clear that any real no. x can be written as

$$x = [x] + \theta \quad \text{for some } 0 \le \theta < 1$$

Next thm provides a way to investigate the number of times a particular prime ~~occurs~~ appears in $n!$ $(n \in \mathbb{N})$ without actually having to ~~write $n!$~~ expand $n!$.

Thm 9 Let $n \in \mathbb{N}$ and p be a prime. Then the exponent (or power) of the highest power of p that divides $n!$ is

$$\sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right]$$

where the series is finite because $[n/p^k] = 0$ for $p^k > n$.

Pf The +ve integers divisible by p and less than or equal to n are

$$p, 2p, 3p, \ldots, tp$$

Observe that tp is the largest integer s.t. $tp \le n$ i.e $t \le n/p$

Hence t is the largest integer s.t. $t \le n/p$ i.e. $t = [n/p]$

Rewriting the list we have.

(1) ... $\quad p, 2p, 3p, \ldots, \left[\frac{n}{p}\right]p$ ← We can also call them multiples of p

Note that this list contains $\left[\frac{n}{p}\right]$ multiples of p

Similarly the integers that are divisible by p^2 and less. n are $p^2, 2p^2, \ldots \left[\frac{n}{p^2}\right]p^2$

Repeating this process finite no. of times and adding the number of multiples of each power of p we get that the number of times p divides $n!$ is

$$\sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right]$$

Example Find the number of zeros with which the decimal representation of 50! terminates

Sol Here actually we have to find the number of times 10 enters into the product 50! Since $10 = 2\times5$ it is enough to find the exponents of 2 and 5 in the prime factorization of 50! and then to select the smaller figure.

Exponent of highest power of 2 ~~d~~ in 50! is

$$\sum_{k=1}^{\infty}\left[\frac{50}{2^k}\right] = \left[\frac{50}{2}\right]+\left[\frac{50}{2^2}\right]+\left[\frac{50}{2^3}\right]+\left[\frac{50}{2^4}\right]+\left[\frac{50}{2^5}\right]+0+0+\cdots$$

$$= 25+12+6+3+1 = 47$$

Similarly the highest power of 5 in 50! is

$$\sum_{k=1}^{\infty}\left[\frac{50}{5^k}\right] = \left[\frac{50}{5}\right]+\left[\frac{50}{5^2}\right]+0+0+\cdots$$

$$= 10+2 = 12$$

$\therefore$ Highest power of 10 in 50! is 12. ~~6~~

$\therefore$ 50! ends with 10 zeros.

Thm 10 If $m, n \in \mathbb{N}$ with $1 \leq n < m$. then the binomial coefficient

$$\binom{m}{n} = \frac{m!}{n!\,(m-n)!}$$

is ~~also~~ an integer.

Pf For any $a, b \in \mathbb{R}$

$[a+b] \geq [a]+[b]$

Hence for each prime factor p of $n!\,(m-n)!$

$$\left.\begin{aligned}&\left[\frac{m-n}{p^k}+\frac{n}{p^k}\right] \geq \left[\frac{m-n}{p^k}\right]+\left[\frac{n}{p^k}\right]\\ \text{i.e. } &\left[\frac{m}{p^k}\right] \geq \left[\frac{m-n}{p^k}\right]+\left[\frac{n}{p^k}\right]\end{aligned}\right\} k=1,2,\ldots$$

$$\Rightarrow \underbrace{\sum_{k=1}^{\infty}\left[\frac{m}{p^k}\right]}_{\text{Highest power of } p \text{ in } m!} \geq \underbrace{\sum_{k=1}^{\infty}\left[\frac{m-n}{p^k}\right]+\sum_{k=1}^{\infty}\left[\frac{n}{p^k}\right]}_{\text{Highest power of } p \text{ in } n!\,(m-n)!}$$

If $a > b$ then $[a] \geq [b]$

$[a] \leq a \quad [b] \leq b$

$\therefore [a]+[b] \leq a+b$

$\Rightarrow [[a]+[b]] \leq [a+b]$

$\Rightarrow [a]+[b] \leq [a+b]$

$\because$ p appears in the numerator and denominator of $\binom{n}{r}$ same no of times.

$\because$ p arbitrary therefore this property holds for every prime divisor of the denominator $r!(n-r)!$

Hence $r!(n-r)! \mid n!$

$\therefore$ $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ is an integer.

__Corollary__ For every $r \in \mathbb{N}$, the product of any r consecutive +ve integers is divisible by $r!$

Pf Let n be the largest of these r consecutive integers. Then their product is

$$n(n-1)(n-2)\cdots(n-r+1)$$
$$= \left[\frac{n!}{r!(n-r)!}\right] \cdot r!$$

Hence $r!$ divides the product

__Thm 11__ Let f and F be number theoretic f-ns s.t

$$F(n) = \sum_{d \mid n} f(d)$$

Then for any +ve integer N

$$\sum_{n=1}^{N} F(n) = \sum_{k=1}^{N} f(k)\left[\frac{N}{k}\right]$$

Pf
$$\sum_{n=1}^{N} F(n) = \sum_{n=1}^{N} \sum_{d \mid n} f(d) = \sum_{d \mid 1} f(d) + \sum_{d \mid 2} f(d) + \cdots + \sum_{d \mid N} f(d)$$

Note that for any k $(1 \le k \le N)$ $f(k)$ occurs in each sum atleast once. We will calculate the no. of sums in which $f(k)$ occurs as a term

[For example if $k=1$ then $f(1)$ occurs in the sum $\sum_{d \mid 1} f(d)$
Also $f(1)$ " " " all other sums
$\therefore$ $f(1)$ occurs in all the sums i.e. N sums]

Note that $f(k)$ occurs in those sums $\left(\sum_{d|n} f(d)\right)$ where $k|n$. Hence it is sufficient to find the number of integers among $1, 2, \ldots, N$ divisible by k. There are exactly $[N/k]$ of them:

$$k, 2k, \ldots [N/k]k$$

$\therefore$ For each k $(1 \leq k \leq N)$ $f(k)$ is a term of the sum $\sum_{d|n} f(d)$ for $[N/k]$ different +ve integers less than or equal to N.

$$\therefore \quad \sum_{n=1}^{N} \sum_{d|n} f(d) = \sum_{k=1}^{N} f(k) \left[\frac{N}{k}\right]$$

<u>Corollary</u> If $N \in \mathbb{N}$, then

(a) $\displaystyle\sum_{n=1}^{N} \tau(n) = \sum_{n=1}^{N} \left[\frac{N}{n}\right]$

(b) $\displaystyle\sum_{n=1}^{N} \sigma(n) = \sum_{n=1}^{N} n \left[\frac{N}{n}\right]$

Pf (a) $f(n) = 1 \quad \forall n$ and $\tau(n) = \sum_{d|n} f(n)$

$$\therefore \sum_{n=1}^{N} \tau(n) = \sum_{n=1}^{N} \sum_{d|n} f(d) = \sum_{n=1}^{N} f(n) \left[\frac{N}{n}\right] = \sum_{n=1}^{N} \left[\frac{N}{n}\right]$$

(b) ll'ly prove

<u>Example</u> $N = 6$

$$\sum_{n=1}^{6} \tau(n) = \tau(1) + \tau(2) + \tau(3) + \tau(4) + \tau(5) + \tau(6)$$
$$= 1 + 2 + 2 + 3 + 2 + 4 = 14$$

$$\sum_{n=1}^{6} \left[\frac{6}{n}\right] = \left[\frac{6}{1}\right] + \left[\frac{6}{2}\right] + \left[\frac{6}{3}\right] + \left[\frac{6}{4}\right] + \left[\frac{6}{5}\right] + \left[\frac{6}{6}\right]$$
$$= 6 + 3 + 2 + 1 + 1 + 1 = 14$$

ll'ly $\displaystyle\sum_{n=1}^{6} \sigma(n) = 33 = \sum_{n=1}^{6} n \left[\frac{6}{n}\right]$