Multiplicative function. A number theoretic function f is said to be multiplicative if
$$f(mn) = f(m)f(n) \quad \text{whenever } \gcd(m,n)=1$$

Thm 3 The functions τ and σ are both multiplicative functions

Pf. Let $m, n \in \mathbb{N}$ s.t. $\gcd(m,n)=1$

Assume $m>1$, $n>1$ (For $m=1$, $n=1$ the result is trivial)

Let m and n have prime factorizations:

$$m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} \quad \text{and} \quad n = q_1^{j_1} q_2^{j_2} \cdots q_s^{j_s}$$

$\because$ $\gcd(m,n)=1$ therefore $p_i \neq q_j$ for any i, j.

$\therefore$ The prime factorization of the product mn is given by

$$mn = p_1^{k_1} \cdots p_r^{k_r} q_1^{j_1} \cdots q_s^{j_s}$$

$$\therefore \tau(mn) = (k_1+1) \cdots (k_r+1)(j_1+1) \cdots (j_s+1)$$
$$= \tau(m)\tau(n)$$

and
$$\sigma(mn) = \frac{p_1^{k_1+1}-1}{p_1-1} \cdots \frac{p_r^{k_r+1}-1}{p_r-1} \cdot \frac{q_1^{j_1+1}-1}{q_1-1} \cdots \frac{q_s^{j_s+1}-1}{q_s-1}$$
$$= \sigma(m)\sigma(n)$$

Lemma 4 If $\gcd(m,n)=1$ then the set of positive divisors of mn consists of all products $d_1 d_2$ where $d_1 | m$, $d_2 | n$ and $\gcd(d_1, d_2)=1$; Furthermore these products are all distinct.

Pf Assume $m>1$ and $n>1$ and

let $m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ and $n = q_1^{j_1} q_2^{j_2} \cdots q_s^{j_s}$

$\because$ $\gcd(m,n)$ therefore the prime factorizations of m and n will not contain any common term i.e. $p_i \neq q_j \ \forall i, j$

Hence the prime factorization of mn is

$$mn = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} q_1^{j_1} q_2^{j_2} \cdots q_s^{j_s}$$

$\therefore$ Any positive divisor of mn can be uniquely written as

$$d = p_1^{a_1} \cdots p_r^{a_r} q_1^{b_1} \cdots q_s^{b_s} \quad 0 \le a_i \le k_i, \quad 0 \le b_i \le j_i$$

Taking $d_1 = p_1^{a_1} \cdots p_s^{a_s}$ and $d_2 = q_1^{b_1} \cdots q_s^{b_s}$ gives us

$d = d_1 d_2$ and clearly $d_1 | m$ and $d_2 | n$

$\because p_i \neq q_j \quad \forall\ i, j$

$\therefore \gcd(d_1, d_2) = 1$

<u>Thm 4</u> If f is a multiplicative function and F is defined by

$$F(n) = \sum_{d|n} f(d)$$

then F is also multiplicative.

Pf Let m, n ~~be prime~~ $\in \mathbb{Z}$ s.t. $\gcd(m, n) = 1$

Then
$$F(mn) = \sum_{d|mn} f(d) = \sum_{\substack{d_1|m \\ d_2|n}} f(d_1 d_2) \quad [\gcd(d_1, d_2) = 1]$$

$$= \sum_{\substack{d_1|m \\ d_2|n}} f(d_1) f(d_2)$$

$$= \sum_{d_1|m} f(d_1) \sum_{d_2|n} f(d_2)$$

$$= F(m) F(n)$$

<u>Example</u> $m = 8$ $n = 3$ $m = 2^3$ $n = 3$

$$F(8 \cdot 3) = \sum_{d|24} f(d)$$

$$= f(1) + f(2) + f(3) + f(4) + f(6) + f(8) + f(12) + f(24)$$

$$= f(1 \cdot 1) + f(2 \cdot 1) + f(1 \cdot 3) + f(4 \cdot 1) + f(2 \cdot 3) + f(8 \cdot 1) + f(4 \cdot 3) + f(8 \cdot 3)$$

$$= f(1)f(1) + f(2)f(1) + f(1)f(3) + f(4)f(1) + f(2)f(3) + f(8)f(1) + f(4)f(3) + f(8)f(3)$$

$$= [f(1) + f(2) + f(4) + f(8)][f(1) + f(3)]$$

c) The Möbius inversion formula

For $n \in \mathbb{N}$ define μ

$$\mu(n) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } p^2 | n \text{ for some prime } p \\ (-1)^r & \text{if } n = p_1 p_2 \cdots p_r \text{ where } p_i \text{ are distinct primes} \end{cases}$$

Möbius μ-function

Example $\mu(1)=1$, $\mu(2)=(-1)^1=-1$

$\mu(3) = (-1)^1 = -1$

$\mu(4) = 0 \quad \because 2^2 | 4$

$\mu(5) = -1 \ldots$

$\mu(6) = (-1)^2 = 1$

Thm 5 μ is a multiplicative function

Pf Let $m, n \in \mathbb{Z}$ s.t. $\gcd(m,n)=1$

Suppose either of m and n are divisible by p^2 i.e. either $p^2 | m$ or $p^2 | n$. ~~Anyway~~ Either way $p^2 | mn$

$\therefore \quad \mu(mn) = 0 = \mu(m)\mu(n)$ [Either $\mu(m)=0$ OR $\mu(n)=0$]

Suppose both m and n are square free. Suppose

$m = p_1 p_2 \cdots p_r$ and $n = q_1 q_2 \cdots q_s$

Note that all the primes p_is and q_is are distinct. Then

$\mu(mn) = (-1)^{r+s} = (-1)^r (-1)^s = \mu(m)\mu(n)$

Thm 6 For each +ve integer $n \geq 1$

$$F(n) = \sum_{d|n} \mu(d) = \begin{cases} 1 & n=1 \\ 0 & n>1 \end{cases}$$

Pf $n=1$ $\quad \sum_{d|1} \mu(d) = \mu(1) = 1$

$n>1$ Firstly, we calculate $F(n)$ for powers of a prime say for $n = p^k$

$$F(n) = F(p^k) = \sum_{d \mid p^k} \mu(d) = \mu(1) + \mu(p) + \mu(p^2) + \cdots + \mu(p^k)$$
$$= \mu(1) + \mu(p) + 0 + 0 + \cdots + 0$$
$$= 1 + (-1) = 0$$

Now let us suppose ~~prop~~ $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$

Then $F(n) = \sum_{d \mid n} \mu(d)$ is a multiplicative f^n [By Thm 4 & $\because \mu$ is multiplicative]

$$\therefore F(n) = F(p_1^{k_1}) F(p_2^{k_2}) \cdots F(p_r^{k_r}) = 0 \cdot 0 \cdots 0 = 0$$

<u>Thm 7</u> (Möbius inversion formula) Let F and f be two number-theoretic f^n related by the formula

$$F(n) = \sum_{d \mid n} f(d)$$

Then
$$f(n) = \sum_{d \mid n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) F(d)$$

Pf We have
$$\sum_{d \mid n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d \mid n} \left(\mu(d) \sum_{c \mid n/d} f(c) \right)$$
$$= \sum_{d \mid n} \sum_{c \mid n/d} \mu(d) f(c)$$

[Now $d \mid n$ and $c \mid n/d \Rightarrow$ $n = d \cdot d'$, $\frac{n}{d} = c \cdot c'$.
$\Leftrightarrow$ $n = d \cdot d'$ and $n = d \cdot c \cdot c' \Leftrightarrow c \mid n$
$\Updownarrow$
$d \mid n/c$]

$$\therefore \sum_{d \mid n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d \mid n} \sum_{c \mid n/d} \mu(d) f(c) = \sum_{c \mid n} \left(\sum_{d \mid n/c} f(c) \mu(d) \right)$$
$$= \sum_{c \mid n} \left(f(c) \sum_{d \mid n/c} \mu(d) \right)$$

Now we have two cases

<u>Case I</u> If $n/c \neq 1$ then $\sum_{d \mid n/c} \mu(d) = 0$ $\Rightarrow$ ~~$\sum_{d \mid n} \mu(d) F\left(\frac{n}{d}\right) = 0$~~

<u>Case II</u> If $n/c = 1$ then $\sum \mu(d) = 1$

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$\therefore \cancel{\sum_{d|n}} \cancel{\int}$

$$\therefore \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{\substack{c|n \\ c=n}} f(c)\cdot 1 = f(n)$$

Now let us replace the dummy index d by $d' = \frac{n}{d} \Rightarrow n = d'd$

$\therefore d'$ is also a positive divisor of n. and

$$f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d'|n} \mu\left(\frac{n}{d'}\right) F(d')$$

<u>Thm 8</u> If F is a multiplicative f^n and

$$F(n) = \sum_{d|n} f(d)$$

then f is also multiplicative

<u>Pf</u> Let $m, n \in \cancel{\mathbb{Z}}$ s.t. $\gcd(m,n) = 1$

~~Then~~ Let d be a divisor of mn. Then we can write d as $d = d_1 d_2$ where $d_1 | m$ and $d_2 | n$ and $\gcd(d_1, d_2) = 1$

Using the inversion formula:

$$f(mn) = \sum_{d|mn} \mu(d) F\left(\frac{mn}{d}\right)$$

$$= \sum_{\substack{d_1|m \\ d_2|n}} \mu(d_1 d_2) F\left(\frac{mn}{d_1 d_2}\right)$$

$$= \sum_{\substack{d_1|m \\ d_2|n}} \mu(d_1 d_2) F\left(\frac{m}{d_1}\right) F\left(\frac{n}{d_2}\right)$$

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$$\Rightarrow f(mn) = \sum_{d_1|m} \mu(d_1) F\left(\frac{m}{d_1}\right) \sum_{d_2|n} \mu(d_2) F\left(\frac{n}{d_2}\right)$$

$$= f(mn)$$