Number theoretic fns

A. The sum and number of divisors

Let $n \in \mathbb{N}$. Then

$\tau(n)$ = number of positive divisors of n

$\sigma(n)$ = sum of all positive divisors of n

Eg) Take $n=6$. It has positive divisors $1, 2, 3, 6$

$\therefore \tau(6)=4$ and $\sigma(6)=1+2+3+6=12$

We will use the following symbol later

$$\sum_{d|n} f(d) \leftarrow \text{Sum of values of } f(d) \text{ where } d \text{ are positive divisors of } n$$

For example $\sum_{d|6} f(d) = f(1)+f(2)+f(3)+f(6)$

Suppose f is the constant fn $f(x)=1$. Then for any $n \in \mathbb{N}$

$$\sum_{d|n} f(d) = f(1)+\cdots+f(n) = 1+\cdots+1 = \tau(n)$$

For example $\sum_{d|6} f(d) = f(1)+f(2)+f(3)+f(6) = 1+1+1+1 = 4 = \tau(6)$

Now let us suppose f is the identity fn $f(x)=x$. Then

$$\sum_{d|n} f(d) = f(1)+\cdots+f(n) = 1+\cdots+n = \sigma(n)$$

For example $\sum_{d|6} f(d) = f(1)+f(2)+f(3)+f(6) = 1+2+3+6 = 12 = \sigma(6)$

In the next thm we will find a way to obtain the positive divisors of any positive integer. To be clear this method is already used by us.

Thm 1 Let $n>1$ be any positive integer with prime factorization $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$. Then the positive divisors of n are precisely those integers 'd' of the form

$$d = p_1^{a_1} \cdot p_2^{a_2} \cdots p_r^{a_r}$$

where $0 \leq a_i \leq k_i$ $(i = 1, 2, \ldots, r)$

Pf If $a_1 = a_2 = \cdots = a_r = 0$ we have $d = p_1^0 p_2^0 \cdots p_r^0 = 1$

If $a_i = k_i$ $\forall$ $i = 1, 2, \ldots, r$ we have $d = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} = n$

Suppose d divides n non-trivially.

Let $n = dd'$ where $d > 1$ and $d' > 1$. We can represent d and d' as product of primes (not necessarily unique) i.e.

$d = q_1 q_2 \cdots q_s$ and $d' = t_1 t_2 \cdots t_u$ q_i and t_j is prime

Then $n = dd' = q_1 q_2 \cdots q_s \cdot t_1 t_2 \cdots t_u$

$\Rightarrow p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} = q_1 q_2 \cdots q_s t_1 \cdots t_u = n$

By uniqueness of the prime factorization each prime q_i must be one of the p_j's. Hence we get (collecting equal primes into a single power)

$$d = q_1 q_2 \cdots q_s = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$$

Conversely ~~suppose~~ every number $d = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ $(0 \leq a_i \leq k_i)$ is a divisor of n since we have

$$n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$$

$$= (p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r})(p_1^{k_1 - a_1} p_2^{k_2 - a_2} \cdots p_r^{k_r - a_r}) \; E$$

$$= d \cdot d'$$

Here $k_i - a_i \geq 0$ $\forall$ i

Thm 2 If $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization of $n > 1$, then

(a) $\tau(n) = (k_1+1)(k_2+1)\cdots(k_r+1)$

(b) $\sigma(n) = \dfrac{p_1^{k_1+1}-1}{p_1-1}\,\dfrac{p_2^{k_2+1}-1}{p_2-1}\cdots\dfrac{p_r^{k_r+1}-1}{p_r-1}$

Pf. (a) Any positive divisor d of n can be written as

$$d = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} \qquad 0 \le a_i \le k_i$$

$\Downarrow$

there are (k_i+1) choices for a_i

i.e. there are (k_1+1) choices for a_1, (k_2+1) choices for a_2 ... ... and (k_r+1) choices for a_r. Therefore there are $(k_1+1)(k_2+1)\cdots(k_r+1)$ ways in which can write d

$\therefore$ The total no. of divisors of n is $\tau(n) = (k_1+1)(k_2+1)\cdots(k_r+1)$

(b) [Lets first understand the idea of the proof that we are going to use through an example.

Let $n = 12$. Then we can write n as $n = 2^2 \cdot 3^1$

Note the product $(1+2+2^2)\cdot(1+3)$. Simplifying it gives

$$= 1+2+2^2+3\cdot1+3\cdot2+3\cdot2^2$$
$$= 1+2+4+3+6+12$$

Clearly this sum contains all +ve divisor of 12 and hence gives the value of $\tau(12)$ which is 28]

Let us consider the product

$$(1+p_1+p_1^2+\cdots+p_1^{k_1})(1+p_2+p_2^2+\cdots+p_2^{k_2})\cdots(1+p_r+p_r^2+\cdots+p_r^{k_r})$$

Expanding this product will give us a sum which contains all +ve divisors of n. Hence

$$\sigma(n) = (1+p_1+p_1^2+\cdots+p_1^{k_1})(1+p_2+p_2^2+\cdots+p_2^{k_2})\cdots(1+p_r+p_r^2+\cdots+p_r^{k_r})$$

$$= \frac{p_1^{k_1+1}-1}{p_1-1}\cdot\frac{p_2^{k_2+1}-1}{p_2-1}\cdots\frac{p_r^{k_r+1}-1}{p_r-1}$$