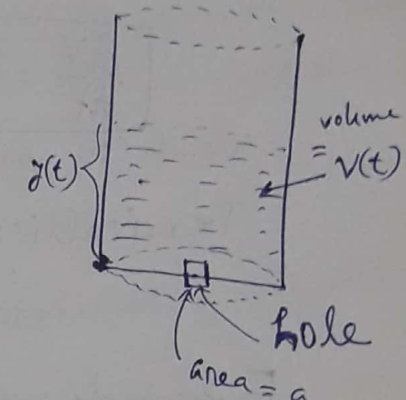
# Mathematical Modelling of Torricell's Law:

**Torricell's Law:** Torricell's Law states that time rate of change of the volume of the water in a draining tank is proportional to the square root of the depth of water in the tank.

We consider a water tank with a bottom hole of area 'a' from which the water is leaking. Let $y(t)$ denotes the depth of water at time t. Let $V(t)$ be the volume of the water in the draining tank. Under ideal conditions, the velocity of water draining through the hole is




$$v = \sqrt{2gy}$$

where $g$ is accl$^n$. due to gravity. It is actually the velocity of a water drop when it falls freely from the surface of the water to the hole.

$$v^2 = u^2 + 2gh$$
$$= 0^2 + 2gy$$
$$\Rightarrow v = \sqrt{2gy}$$
$(\because \text{ here } h = y)$

Then,

$$\frac{dV}{dt} = -av = -a\sqrt{2gy} \rightarrow (1)$$

or, $$\frac{dV}{dt} = -\lambda\sqrt{y}, \quad \text{where } \lambda = a\sqrt{2g} \rightarrow (2)$$

i.e. $$\frac{dV}{dt} \propto \sqrt{y}$$

which is Torricelli's law.

Again, let $A(y)$ be the horizontal cross-sectional area of the tank at a height $y$ above the hole. Then

$$V = \int_0^y A(y)\,dy \rightarrow (*)$$

$\therefore$ By fundamental theorem of calculus, from $(*)$,

$$\frac{dV}{dy} = A(y)$$

$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dy}\cdot\frac{dy}{dt} = A(y)\frac{dy}{dt} \longrightarrow (3)$$

$\therefore$ From (1), (2) and (3), we get

$$\boxed{\frac{dV}{dt} = A(y)\frac{dy}{dt} = -a\sqrt{2gy} = -\lambda\sqrt{y}} \rightarrow (4)$$

which is mathematical model (diff. eqn.) of Torricelli's Law.

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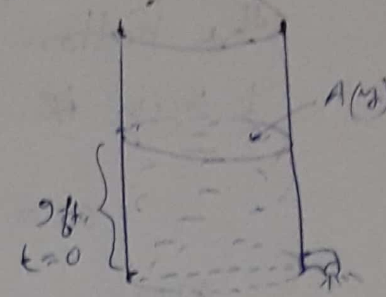
Example:→ (i) A tank ~~has~~ is shaped like a vertical cylinder; it initially contains water to a depth of 9 ft, and a bottom plug is removed at $t=0$ hours. After 1 hour, the depth of the water has dropped to 4 ft. How long does it take for all the water to drain from the tank? (ii) Suppose that the tank has a radius of 3 ft and that its bottom hole is circular, with radius 1 inch. How long it will take the water (initially 9 ft. deep) to drain completely?

Sol: (i) According to Torricelli's law, if $y(t)$ denotes the depth of water in the tank $t$, then




$$A(y)\frac{dy}{dt} = -\lambda\sqrt{y}, \quad \rightarrow (1)$$

where $A(y)$ denotes the horizontal cross-sectional area of the tank at a height $y$ ~~depth~~ above the plug.

from (1),

$$\frac{dy}{dt} = -\frac{\lambda}{A(y)}\sqrt{y}$$

$$\Rightarrow \frac{dy}{dt} = -h\sqrt{y}, \text{ where } h = \frac{\lambda}{A(y)}, \text{ since } A(y) \text{ is const. for cylinder.}$$

$$\Rightarrow \frac{dy}{\sqrt{y}} = -h\,dt$$

Integrating

$$2\sqrt{y} = -ht + c_1, \quad c_1 \rightarrow \text{const. of integration.}$$

$$\rightarrow (2)$$

Initially,

$$y = 9 \text{ at } t = 0.$$

$$\therefore (2) \Rightarrow 2\sqrt{9} = -h\cdot 0 + c_1 \Rightarrow c_1 = 6.$$

$$\therefore (2) \Rightarrow 2\sqrt{y} = -ht + 6; \quad \rightarrow (3)$$

Again, given, $y = 4$, when $t = 1$.

Putting this in (3), we get—

$$2\sqrt{4} = -h\cdot 1 + 6 \Rightarrow h = 2.$$

Putting $h = 2$ in (3),

$$2\sqrt{y} = -2t + 6 \longrightarrow (4)$$

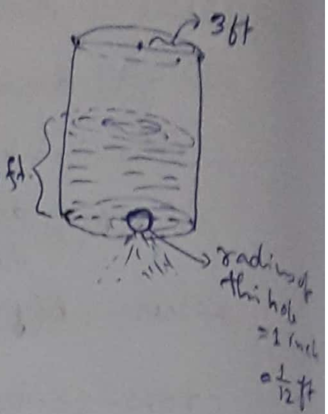
In order to determine $t$ where all the water get drain from the tank, we put $y = 0$ and find $t$.

$$\therefore 2\sqrt{0} = -2t + 6$$

$$\Rightarrow t = 3.$$

$\therefore$ The tank gets empty in 3 hours.

(ii) Let 'a' be the area of the bottom hole of radius 1 inch i.e. 1/12 ft. Then




$$a = \pi\left(\frac{1}{12}\right)^2$$

Also,

$$A(y) = \pi(3)^2 = 9\pi.$$

We know from Torricelli's law,

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy}, \text{ where } g = 32 \text{ ft/sec}^2$$

$$\Rightarrow 9\pi \frac{dy}{dt} = -\pi\left(\frac{1}{12}\right)^2 \sqrt{2\times 32y}$$

$$\Rightarrow 9\pi \frac{dy}{dt} = -\frac{\pi}{\cancel{12}\times\cancel{12}} \times \cancel{8}\sqrt{y}$$

$$\Rightarrow 162 \frac{dy}{dt} = -\sqrt{y}$$

$$\Rightarrow 162 \frac{dy}{\sqrt{y}} = -dt$$

Integrating,

$$162\times 2\sqrt{y} = -t + c_2 \rightarrow (5)$$

Initially, $t = 0$, when $y = 9$.

$$\therefore (5) \Rightarrow 324 \times 3 = c_2$$

$$\Rightarrow c_2 = 972$$

$$\therefore (5) \Rightarrow 324\sqrt{y} = -t + 972 \rightarrow (6)$$

Now to find $t$ when $y = 0$. Putting $y = 0$ in (6), we get

$$t = 972.$$

So, the tank will take 972 seconds to drain completely.

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