Ex. Obtain Lagrange's Interpolation formula in the form

$$f(x) = \sum_{r=0}^{n} \frac{\phi(x) f(x_r)}{(x-x_r)\,\phi'(x_r)}$$

where $\phi(x) = (x-x_0)(x-x_1) \cdots (x-x_n)$

Solⁿ. First we prove Lagrange's Int. formula as

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f(x_0)$$

$$+ \cdots\cdots$$

$$+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} f(x_n)$$

$$\longrightarrow (1)$$

Now, we are given

$$\phi(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$$

$$\Rightarrow \log \phi(x) = \log(x-x_0) + \log(x-x_1) + \cdots + \log(x-x_n)$$

Differentiating w.r.t. x, we get

$$\frac{\phi'(x)}{\phi(x)} = \frac{1}{x-x_0} + \frac{1}{x-x_1} + \cdots + \frac{1}{x-x_n}$$

$$\Rightarrow \phi'(x) = \frac{\phi(x)}{x-x_0} + \frac{\phi(x)}{x-x_1} + \cdots + \frac{\phi(x)}{x-x_n}$$

$$= (x-x_1)(x-x_2)+\cdots+(x-x_n)$$
$$+ (x-x_0)(x-x_2)\cdots(x-x_n) + \cdots$$
$$+ \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

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$$\therefore \phi'(x_0) = (x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)$$

$$\phi'(x_1) = (x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)$$

$$\cdots\cdots\cdots$$

$$\phi'(x_n) = (x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1}).$$

$$\therefore (1) \Rightarrow f(x) = \frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n)}{(x-x_0)(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f(x_0)$$

$$+ \cdots\cdots$$

$$+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})(x-x_n)}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})(x-x_n)} f(x_n)$$

$$\Rightarrow f(x) = \frac{\phi(x)}{(x-x_0)\phi'(x_0)} f(x_0) + \frac{\phi(x)}{(x-x_1)\phi'(x_1)} f(x_1)$$

$$+ \cdots + \frac{\phi(x)}{(x-x_n)\phi'(x_n)} f(x_n)$$

$$= \sum_{r=0}^{n} \frac{\phi(x)}{(x-x_r)\phi'(x_r)} f(x_r)$$