Ex. Show that the n^{th} divided difference for the arguments $x_0, x_1, x_2, \cdots, x_n$ of the function $f(x) = \frac{1}{x}$ is $\frac{(-1)^n}{x_0 x_1 \cdots x_n}$.

Soln. We have, $f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\frac{1}{x_1} - \frac{1}{x_0}}{x_1 - x_0}$

$$= \frac{x_0 - x_1}{x_1 x_0 (x_1 - x_0)} = \frac{-1}{x_0 x_1} = \frac{(-1)^1}{x_0 x_1}$$

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} = \frac{-\frac{1}{x_1 x_2} + \frac{1}{x_0 x_1}}{x_2 - x_0}$$

$$= \frac{x_2 - x_0}{x_0 x_1 x_2 (x_2 - x_0)} = \frac{1}{x_0 x_1 x_2} = \frac{(-1)^2}{x_0 x_1 x_2}$$

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0} = \frac{\frac{1}{x_1 x_2 x_3} - \frac{1}{x_0 x_1 x_2}}{x_3 - x_0}$$

$$= \frac{x_0 - x_3}{x_0 x_1 x_2 x_3 (x_3 - x_0)} = \frac{-1}{x_0 x_1 x_2 x_3}$$

$$= \frac{(-1)^3}{x_0 x_1 x_2 x_3}$$

Proceeding in this way, we get

$$f(x_0, x_1, x_2, \cdots, x_n) = \frac{(-1)^n}{x_0 x_1 x_2 \cdots x_n}$$

#

<u>Theorem</u> :→ The divided difference are symmetrical in all their arguments, i.e. the value of any difference is independent of the order of the arguments.

<u>Proof</u> :→ We have

$$\delta(x_1, x_0) = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_0 - y_1}{x_0 - x_1} = \delta(x_0, x_1) \rightarrow (i)$$

$f(x_1, x_0) = \delta(x_1, x_0)$

Also,

$$\delta(x_1, x_0) = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1}{x_1 - x_0} - \frac{y_0}{x_1 - x_0} =$$

$$= \frac{y_1}{x_1 - x_0} + \frac{y_0}{x_0 - x_1}$$

Again,

$$\delta(x_2, x_1, x_0) = \frac{\delta(x_2, x_1) - \delta(x_1 - x_0)}{x_2 - x_0}$$

$$= \frac{1}{x_2 - x_0}\left[\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}\right]$$

$$= \frac{1}{x_2 - x_0}\left[\frac{y_2}{x_2 - x_1} - \frac{y_1}{x_2 - x_1} - \frac{y_1}{x_1 - x_0} + \frac{y_0}{x_1 - x_0}\right]$$

$$= \frac{1}{x_2 - x_0}\left[\frac{y_2}{x_2 - x_1} + \frac{y_1(x_2 - x_0)}{(x_1 - x_2)(x_1 - x_0)} + \frac{y_0}{x_1 - x_0}\right]$$

$$= \frac{y_2}{(x_2 - x_0)(x_2 - x_1)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_0}{(x_0 - x_1)(x_0 - x_2)}$$

By (i),

$$\delta(x_2, x_1, x_0) = \delta(x_0, x_1, x_2)$$

Similarly,

$$\delta(x_3, x_2, x_1, x_0) = \frac{y_3}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} + \frac{y_2}{(x_2 - x_3)(x_2 - x_1)(x_2 - x_0)}$$

$$+ \frac{y_1}{(x_1 - x_3)(x_1 - x_2)(x_1 - x_0)} + \frac{y_0}{(x_0 - x_3)(x_0 - x_2)(x_0 - x_1)}$$

The right side members of the above eqns remain unchanged when any two values of x are interchanged and the corresponding values of y are also interchanged. This means that a divided difference remain unchanged regardless of how much its arguments are interchanged. We may therefore write

$$\delta(x_n, x_{n-1}, \cdots, x_1, x_0) = \delta(x_0, x_1, \cdots, x_n).$$

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§ Relation between divided difference & ordinary difference:

We have $\delta(x_1, x_0) = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{\Delta y_0}{h}$

when the arguments are equally spaced.

Now,

$$\delta(x_2, x_1, x_0) = \frac{\delta(x_2 - x_1) - \delta(x_1, x_0)}{x_2 - x_0}$$

$$= \frac{1}{x_2 - x_0}\left[\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}\right]$$

$$= \frac{1}{2h}\left[\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right]$$

$$= \frac{1}{2h^2}\left[\Delta y_1 - \Delta y_0\right]$$

$$= \frac{1}{2h^2}\Delta^2 y_0$$

Similarly, $\delta(x_3, x_2, x_1, x_0) = \dfrac{\Delta^3 y_0}{3!\,h^3}$

In general, $\delta^n y_k = \delta(x_k + nh, \cdots, x_k + h, x_k)$

$$= \frac{\Delta^n y_k}{n!\,h^n}, \quad k = 0, 1, 2, \cdots$$

when the independent variables are equally spaced and h is the length of the interval.

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