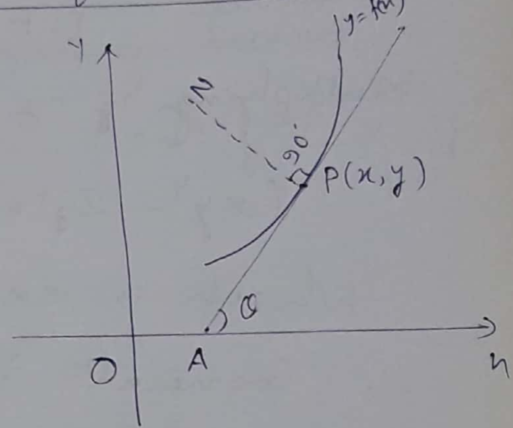
Differential Equations and Mathematical Model

In this section we shall formulate some problems of co-ordinate geometry, or, of some natural laws or phenomenon, mathematically to obtain a differential eqn. (which is also called the mathematical model of the problem or of the law or phenomenon). Then we solve the differential eqn.

1. Application in Coordinate Geometry :

Geometrical interpretation of derivative:

The derivative $\frac{dy}{dx}$ represents the slope of the tangent at (x, y) to the curve $y = f(x)$.



Thus,

$$m = \tan\theta = \frac{dy}{dx}$$

Hence the slope of the normal at (x, y) to the curve $y = f(x)$ is

$$= -\frac{1}{\frac{dy}{dx}}$$

Note: We have learnt the above conception in H.S. course

Ex1. In the following problems, a function $y = h(x)$ is described by some geometric property of its graph. Write a differential equation of the form $\frac{dy}{dx} = f(x,y)$ having the function h as its solution.

(i) Every straight line normal to the graph of h passes through the point $(0,1)$.

(ii) The line tangent to the graph of h at (x,y) passes through the point $(-y, x)$

(iii) The graph of h is normal to every curve of the form $y = x^2+k$, where k is a constant, where they meet.

Solⁿ. (i) We know that the eqn. of the straight line passing through the point $(0,1)$ and slope 'm' is

$$(y-1) = m(x-0)$$

here, $m = \dfrac{-1}{\frac{dy}{dx}}$

Eqn. of line passing through (x_0, y_0) and having slope m is $(y-y_0) = m(x-x_0)$

$$\Rightarrow y-1 = \frac{-1}{\frac{dy}{dx}}(x-0)$$

$$\Rightarrow (y-1)\frac{dy}{dx} = -x$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y-1} \quad \leftarrow \text{Ans.}$$

(ii) Eqn. of the tangent with slope $\frac{dy}{dx}$ and passing through the point $(-y, x)$ is

$$(y-x) = \frac{dy}{dx}(x+y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y-x}{y+x} \quad \longleftarrow \text{Ans.}$$

(iii) Slope of the tangent to the graph h is

$$m_1 = \frac{dy}{dx}.$$

Again, slope of the tangent to the curve $y = x^2+k$ is

$$m_2 = \frac{d}{dx}(x^2+k)$$
$$= 2x.$$

According to question, graph of h is normal to every curve $y = x^2+k$.

$\therefore$ The above two tangents must be perpendicular to each other. So by the condition of orthogonality, we get,

$$m_1 m_2 = -1$$

$$\Rightarrow \frac{dy}{dx} \cdot 2x = -1$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{2x} \quad \leftarrow \text{Ans.}$$

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② Applications of Differential Equations in Science and Engineering :→

Math model of Newton's second Law of motion:

The Law is: The time rate of change of momentum of a body proportional to the resultant force acting on the body and is in the direction of this resultant force

Mathematical form (or Mathematical model) of the law

We know that $\frac{dy}{dx}$ represents the rate of change of y with respect to x. (We found it in HS Course)

Now, let m be the mass of the body and F be the resultant force acting on the body. Let v be the velocity (actually magnitude of the vel.) of the body at any time 't' in the direction of the force.

$\therefore$ momentum $= mv$

& rate of change of momentum $= \frac{d}{dt}(mv)$

$\therefore$ By Newton's 2^{nd} law,

$$\frac{d}{dt}(mv) \propto F$$

or, $\frac{d}{dt}(mv) = kF$, where k is the constant of porportionality. $\longrightarrow (1)$

or, $m\frac{dv}{dt} = kF$, if m is constant.

or, $ma = kF$, where $a = \frac{dv}{dt} = $ accl^n.

or, $F = \frac{1}{k} ma$

or, $\boxed{F = ma}$, if $k=1$

$\longrightarrow (2)$

Which is the mathematical form (or, mathematical model) of Newton's 2^{nd} law of motion.

Note: (1) & (2) actually are differential Eqns. //