Gauss's theorem for a volume distribution of charge :—

If we have a volume charge density ρ. Then the total charge in a volume space V is given by

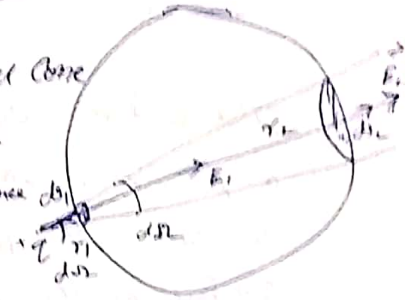
$$q = \iiint_V \rho dv$$

Then the Gauss's theorem can be stated as

$$\phi_E = \iint \vec{E} \cdot d\vec{s} = \frac{q}{\epsilon_0} = \frac{1}{\epsilon_0} \iiint_V \rho dv.$$

Q. Show that electric flux through the surface of a sphere due to a point charge lying outside it is zero.

Ans: Suppose a point charge $+q$ lies outside a sphere as shown. From the point charge draw a small cone making a solid angle $d\Omega$. This cone cuts off a surface area $d\vec{s}_1$, on the nearer side at a distance r_1 and an area $d\vec{s}_2$ on the farther side at a distance r_2



The electric flux through $d\vec{s}_1 = d\phi_1 = \vec{E}_1 \cdot d\vec{s}_1$

where $\vec{E}_1$ is the electric field at ds_1 due to the charge $+q$. It is clear from the figure that $\vec{E}_1$ and $d\vec{s}_1$ point in opposite direction, ie the angle between them $= 180° = \pi$. $\longrightarrow$ (i)

$$\therefore d\phi_1 = \vec{E}_1 \cdot d\vec{s}_1 = E_1 ds_1 \cos 180° = -E_1 ds_1$$

The magnitude of E_1 at ds_1 is $E_1 = \frac{q}{4\pi\epsilon_0 r_1^2}$

$$\therefore d\phi_1 = -\frac{q}{4\pi\epsilon_0 r_1^2} ds_1 \quad \text{——— (ii)}$$

Now $\frac{ds_1}{r_1^2} = d\Omega$ = solid angle subtended by ds_1 at the charge q.

$$\therefore d\phi_1 = -\frac{q}{4\pi\epsilon_0} d\Omega \quad \text{———} \rightarrow \text{(iii)}$$

Similarly electric flux through $d\vec{s}_2 = d\phi_2 = \vec{E}_2 \cdot d\vec{s}_2$

where E_2 is the electric field at ds_2 due to charge $+q$. It is clear from the fig. that E_2 and ds_2 in the same direction, ie. the angle between them is zero.

$\rightarrow$

C

$\therefore\ d\phi_2 = \vec{E_2}.\vec{ds_2} = E_2\, ds_2 \cos 0 = E_2\, ds_2$

The magnitude of $\vec{E_2}$ is

$$\vec{E_2} = \frac{q}{4\pi\epsilon_0 r_2^2}$$

$$\therefore\ d\phi_2 = \frac{q}{4\pi\epsilon_0 r_2^2}\, ds_2$$

But $\frac{ds_2}{r_2^2} = d\Omega$ = solid angle subtended by ds_2 at charge q.

$$\therefore\ d\phi_2 = \frac{q}{4\pi\epsilon_0}\, d\Omega .$$

$\therefore$ The total electric flux through the surface ds_1 and ds_2 is

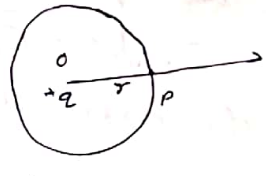
$$d\phi = d\phi_1 + d\phi_2$$

$$= \frac{-q}{4\pi\epsilon_0}\, d\Omega + \frac{q}{4\pi\epsilon_0}\, d\Omega = 0$$

The entire surface of the sphere can be divided in to such a pair of areas. Each pair will make zero contribution towards electric flux. Hence total electric flux through the surface of the sphere due to a point charge lying out side it is zero.

APPLICATION OF GAUSS'S LAW

1. Proof of Coulomb's law.
(Electric intensity due to a point charge)



Let P be a point situated at a distance r from O where a charge +q is located. Imagine a sphere of radius r and having its centre O, point P will be on the surface of such a sphere. Electric intensity E is normal to the surface of the sphere at all points.

$$\therefore\ \oint \vec{E}.\vec{ds} = \frac{1}{\epsilon_0} q$$

$$= E \times 4\pi r^2 = \frac{q}{\epsilon_0}$$

$$\text{or}\quad E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

which is the required value. If a charge q_0 is placed

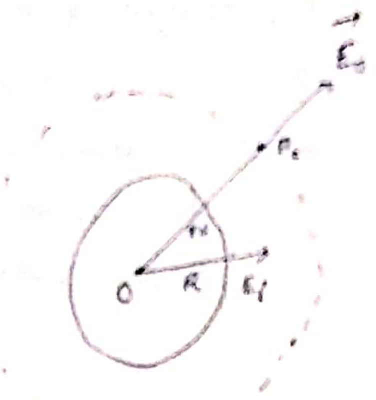
$\rightarrow$

at P then the force acting on it due to the charge q is

$$F = E \cdot q_0 = \frac{1}{4\pi\epsilon_r}\frac{q \cdot q_0}{r^2}$$ which is the Coulomb's law.

2. Electric field due to spherically symmetric charge distribution.

A spherically symmetric charge distribution is said to be the one in which the charge density $\rho(r)$ is function of the distance from the centre and does not depend upon direction.



Consider a sphere of radius R having charge uniformly distributed inside it.

Case-I point lying outside.

Let P_0 be the point situated at a distance r_0 $(r_0 > R)$ where the field strength is to be calculated. Imagine a concentric sphere of radius r_0 drawn such that p lies on the surface of such a sphere. If E_0 be the strength of the electric field at P_0 then

$$\oint \vec{E_0} \cdot \vec{ds} = E_0 4\pi r_0^2$$

Total charge inside the sphere $= \iiint_V \rho(r)\, dv = q$

According to Gauss's theorem

$$\oint \vec{E_0} \cdot \vec{ds} = \frac{1}{\epsilon_0 \epsilon_r}\iiint_V \rho(r)\, dv$$

$$\therefore E_0 4\pi r_0^2 = \frac{1}{\epsilon_0\epsilon_r} q$$

$$\therefore E_0 = \frac{1}{4\pi\epsilon_0\epsilon_r}\frac{q}{r_0^2} \quad \text{--- (1)}$$

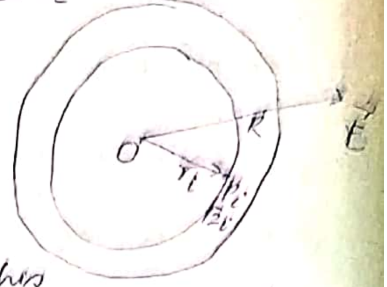
This expression is the same if the whole of the charge were concentrated at the centre O.

Case II - point lying on the surface of the sphere:-

When the point P is lying on the surface of the sphere then $r_0 = R$, then we have $E_s = \frac{1}{4\pi\epsilon_0\epsilon_r}\frac{q}{R^2}$ --- (2)

Case C :- point P lying inside the sphere :-

Let the point P be situated at distance r_i ($r_i < R$) from the centre O. Imagine a Concentric sphere of radius r_i such that the point P_i lies on the surface of the sphere. A part of the charge distribution lies out of this sphere. This charge will not produce any electric flux through the sphere. The electric flux linked with this sphere is due to charge contained inside it. If E_i is the electric intensity at P_i



$$\oint \vec{E_i} \cdot \vec{ds} = E_i 4\pi r_i^2$$

Let Q = Total charge inside the spherical charge distribution $\rho = \frac{\text{charge}}{\text{volume}} = \frac{Q}{\frac{4}{3}\pi R^3} = \frac{3Q}{4\pi R^3}$

∴ Charge inside the sphere of radius r_i

$= \rho \times$ volume of the sphere

$$= \frac{3Q}{4\pi R^3} \times \frac{4}{3}\pi r_i^3 = \frac{Q r_i^3}{R^3}$$

According to Gauss's theorem

$$\oint \vec{E_i} \cdot \vec{ds} = \frac{1}{\epsilon_r \epsilon_0} \times \text{total charge inside the sphere}$$

$$\Rightarrow E_i \times 4\pi r_i^2 = \frac{1}{\epsilon_r \epsilon_0} \frac{Q r_i^3}{R^3}$$

$$\Rightarrow E_i = \frac{1}{4\pi \epsilon_0 \epsilon_r} \frac{Q r_i}{R^3} \quad \text{——— (3)}$$

It is clear from eqn (3) that

$$E_i \propto r_i$$

A graphical view of electric field against distance from the centre of the spherical charge distribution is shown in the fig.