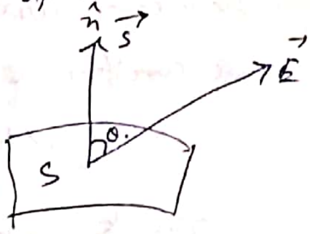
Electric Flux and electric flux density.

Electric Flux:- For a uniform electric field the electric flux ϕ_E through an area s is defined as the product of the area and the component of electric field vector $\vec{E}$ normal to the area.

$$\phi_E = ES\cos\theta.$$

where θ is the angle between the direction of $\vec{E}$ and positive normal to area s.



If s is the magnitude of the area vector $\vec{S}$ and $\hat{n}$ a unit vector in the direction of positive normal, then $\vec{S} = s\hat{n}$

Therefore the electric flux can be written as the ~~sct~~ scalar product of two vectors.

$$\phi_E = \vec{E}\cdot\vec{S} = \vec{E}\cdot s\hat{n}$$

For a closed surface the direction of $\hat{n}$ is along the outward drawn normal to the surface.

For a non-uniform electric field, ie if $\vec{S}$ is very large or $\vec{E}$ varies from point to point, the area s is dividing to small elementary area Δs, Then the electric flux over the elementary area Δs is given by $\Delta\phi_E = \vec{E}\cdot\Delta\vec{S}$ $\longrightarrow$ (i)

$\therefore$ The electric flux over the whole area

$$\phi_E = \sum_{s} \vec{E}\cdot\Delta\vec{S}.$$

As $\Delta s \to 0$ we can replace ~~write~~ $\Delta\vec{S}$ by ~~an~~ $d\vec{S}$ and $\Delta\phi_E$ by $d\phi_E$ in (i)

$$d\phi_E = \vec{E}\cdot d\vec{S}. \longrightarrow (ii)$$

$$\therefore \phi_E = \iint_s \vec{E}\cdot d\vec{S}.$$

where $\iint_s$ represents the surface integral over the area $\vec{S}$.

$\longrightarrow$

(2)

DIMENSION AND UNITS OF ELECTRIC FLUX :-

The SI unit of electric flux is $N m^2 C^{-1}$ and the dimension of electric flux ϕ_E $= [ML^3 T^{-3} A^{-1}]$

ELECTRIC FLUX DENSITY

The electric flux density is defined as electric flux per unit area. If ϕ_E is electric flux through area A in a uniform electric field then

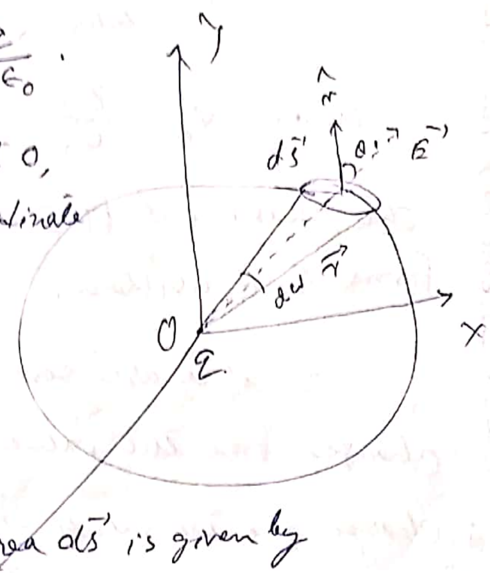
$$\text{Electric flux density } E = \frac{\phi_E}{A}.$$

The SI unit of electric flux density is N/C

GAUSS'S THEOREM IN ELECTROSTATIC.

It states that the total electric flux in free space (or vacuum) through a closed surface is equal to $\frac{1}{\epsilon_0}$ times the charges enclosed by the surface.

$$\text{i.e.}\quad \phi_s = \oint\!\!\oint_s \vec{E}\cdot d\vec{s} = \frac{q}{\epsilon_0}.$$



Consider a charge q situated at O, the origin of the rectangular co-ordinate system. Let S be the Gaussian surface around this charge.
Now consider a small elementary area $d\vec{s}$ at a vector distance $\vec{r}$ from the charge q.
The electric flux through the area $d\vec{s}$ is given by

$$d\phi_E = \vec{E}\cdot d\vec{s} \quad \text{where } \vec{E} \text{ is the electric field vector}$$

at $\vec{r}$.
The total electric flux over the (closed) Gaussian surface due to the charge q inside it, in free space is given by

$$\phi_E = \int d\phi_E = \oint\!\!\oint \vec{E}\cdot d\vec{s}$$

Now the electric intensity $\vec{E}$ at a point on the elementary surface $d\vec{s}$ at the position vector $\vec{r}$ due to the charge q at the origin is given by $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$. where $\hat{r}$ is the unit vector in the direction of $\vec{E}$. Also $d\vec{s} = ds\,\hat{n}$, where $\hat{n}$ is the unit vector in the direction of outward drawn normal to the surface ds.

$$\therefore \phi_E = \frac{1}{4\pi\epsilon_0}\oint\!\!\oint \frac{q}{r^2}\hat{r}\cdot\hat{n}\,ds$$

$$= \frac{1}{4\pi\epsilon_0}\oint\!\!\oint \frac{q}{r^2}\,ds\cos\theta \qquad (\hat{r}\cdot\hat{n} = \cos\theta)$$

$$= \frac{1}{4\pi\epsilon_0}\oint\!\!\oint q\,\frac{ds\cos\theta}{r^2}.$$

Now $\frac{ds\cos\theta}{r^2} = d\omega$, the small solid angle subtended by the elementary surface $d\vec{s}$ at q.

$$\therefore \phi_E = \frac{1}{4\pi\epsilon_0}\oiint q \cdot d\omega = \frac{q}{4\pi\epsilon_0}\oiint d\omega$$

$$\text{or } \phi_E = \frac{q}{4\pi\epsilon_0}\omega = \frac{q}{4\pi\epsilon_0}4\pi = \frac{q}{\epsilon_0}$$

Hence $\phi_E = \frac{q}{\epsilon_0}$ where $\phi_E = \oiint \vec{E}\cdot d\vec{s}$;

The statement $\oiint \vec{E}\cdot d\vec{s} = \frac{q}{\epsilon_0}$ is known as integral form of Gauss's law.

If there are several charges inside, the positive charges give the values $\frac{q_1}{\epsilon_0} + \frac{q_2}{\epsilon_0} + \cdots$ and the negative charges give the values $-\frac{q_1'}{\epsilon_0} - \frac{q_2'}{\epsilon_0} - \cdots$, the total electric flux due to all the charges $= \frac{1}{\epsilon_0}(q_1 + q_2 + \cdots - q_1' - q_2' \cdots)$. Thus the total charge inside the surface refers to the algebraic sum of the charges.

Therefore for a number of point charges within the closed surface, Gauss's theorem can be stated as

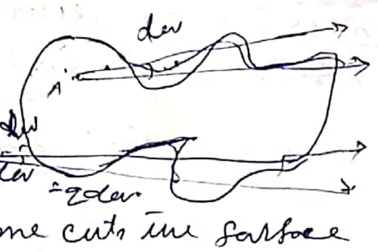
$$\phi_s = \oiint \vec{E}\cdot d\vec{s} = \frac{1}{\epsilon_0}\Sigma q_i$$

Instead of free space if we consider a dielectric of permittivity ϵ_r then Gauss's law becomes: $\phi = \oiint \vec{E}\cdot d\vec{s} = \frac{1}{\epsilon_0\epsilon_r}\Sigma q_i$

When the charge lies outside the surface:-

If the charge lies outside the surface the electric flux inwards is equal to the electric flux outwards and the resultant flux for the whole surface is zero.

When the charge q lies inside as at A a small cone cuts the surface an odd numbers of times so that electric flux for the cone at A $= +q\,d\omega - q\,d\omega + q\,d\omega - q\,d\omega + q\,d\omega$



If the charge lies outside at B, the cone cuts the surface an even number of times and the contribution of any cone $= q\,d\omega - q\,d\omega + q\,d\omega - q\,d\omega = 0$

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Gauss's theorem for a volume distribution of charge :—

If we have a volume charge density ρ. Then the total charge in a volume space V is given by

$$q = \iiint_V \rho \, dv$$

Then the Gauss's theorem can be stated as

$$\phi_E = \iint \vec{E} \cdot d\vec{s} = \frac{q}{\epsilon_0} = \frac{1}{\epsilon_0} \iiint_V \rho \, dv.$$