Example of Rule – III

Rule - III: If $Mdx + Ndy = 0$ is homo & $Mx + Ny \neq 0$, then $\frac{1}{Mx+Ny}$ is an I.F.

Ex. 1. Solve:

$$x^2y\,dx - (x^3+y^3)dy = 0$$

Solⁿ: The given eqn. is homogeneous.

H.W. Try to solve this eqn. by putting $y = vx$

Also here

$$M = x^2y, \quad N = -(x^3+y^3)$$

$$\therefore \frac{\partial M}{\partial y} = x^2, \quad \frac{\partial N}{\partial x} = -3x^2$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$, so that the eqn. is not exact.

Here, we see that

$$Mx + Ny = x^3y - x^3y - y^4 = -y^4 \neq 0$$

$$\therefore \text{An I.F.} = \frac{1}{Mx+Ny} = -\frac{1}{y^4}$$

∴ Multiplying by $-\frac{1}{y^4}$, the given eqn. becomes

$$-\frac{x^2}{y^3}dx + \frac{x^3+y^3}{y^4}dy = 0$$, which is exact now.

∴ General solⁿ is

$$\int_{y\text{-const}} \left(-\frac{x^2}{y^3}\right)dx + \int \frac{1}{y}\,dy = \log C$$

$$\Rightarrow -\frac{x^3}{3y^3} + \log y = \log C$$

$$\Rightarrow \log\left(\frac{y}{C}\right) = \frac{x^3}{3y^3}$$

$$\Rightarrow \frac{y}{C} = e^{x^3/3y^3}$$

$$\Rightarrow y = Ce^{x^3/3y^3}$$ ← Ans

Example of Rule-IV

Ex. Solve

$$y(xy+2x^2y^2)dx + x(xy - x^2y^2)dx = 0$$

Sol^n Given eqn. is of the form

$$y\,f(xy)\,dx + x\,g(xy)\,dy = 0$$

where, $f(xy) \neq g(xy)$.

$$\therefore \text{I.F.} = \frac{1}{Mx - Ny}$$

$$= \frac{1}{xy(xy+2x^2y^2) - xy(xy - x^2y^2)}$$

$$= \frac{1}{xy\{xy + 2x^2y^2 - xy + x^2y^2\}}$$

$$= \frac{1}{3x^3y^3}$$

$\therefore$ Multiplying by $\frac{1}{3x^3y^3}$, the given eqn becomes

$$\left(\frac{xy^2 + 2x^2y^3}{3x^3y^3}\right)dx + \frac{x^2y - x^3y^2}{3x^3y^3}\,dy = 0$$

$$\Rightarrow \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0$$

which is exact now.

$\therefore$ General sol^n is

$$\int_{y=\text{const}} \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int\left(-\frac{1}{3y}\right)dy = \log C$$

$$\Rightarrow -\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = \log c_1$$

$$\Rightarrow -\frac{1}{xy} + \log x^2 - \log y = 3\log c_1 = \log c$$

$$\Rightarrow \log\frac{x^2}{yc} = \frac{1}{xy} \Rightarrow \frac{x^2}{yc} = e^{1/xy} \Rightarrow \frac{x^2}{y} = c\,e^{1/xy}$$

Example of Rule - V

Ex. Solve $$(y^3 - 2yx^2)dx + (2xy^2 - x^3)dy = 0 \to (1)$$

Sol^n. The eqn. can be written in the form

$$y^2(y\,dx + 2x\,dy) - x^2(2y\,dx + x\,dy) = 0$$

Which is of the form

$$x^a y^b(my\,dx + nx\,dy) + x^c y^d(\mu y + \nu x\,dy) = 0$$

Now let $x^\alpha y^\beta$ be an integrating factor of the eqn

Multiplying by $x^\alpha y^\beta$, the given eqn. becomes

$$\left(y^{3+\beta}x^\alpha - 2y^{\beta+1}x^{2+\alpha}\right)dx + \left(2x^{1+\alpha}y^{2+\beta} - x^{\alpha+3}y^\beta\right)dy = 0$$

In this exact eqn,

$$M = y^{3+\beta}x^\alpha - 2y^{\beta+1}x^{2+\alpha}$$

$$N = 2x^{1+\alpha}y^{2+\beta} - x^{\alpha+3}y^\beta$$

Hence α and β are such that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

i.e. $$(3+\beta)y^{2+\beta}\cdot x^\alpha - 2(\beta+1)y^\beta x^{2+\alpha}$$
$$= 2(1+\alpha)x^\alpha y^{2+\beta} - (\alpha+3)x^{\alpha+2}y^\beta$$

Equating the co-efficients of $y^{2+\beta}x^\alpha$ and $y^\beta x^{2+\alpha}$,

$$3+\beta = 2(1+\alpha) \to (2)$$

& $$2(\beta+1) = \alpha+3. \to (3)$$

Scanned with CamScanner

Now,

$(2) \Rightarrow \beta = -3+2+2\alpha = -1+2\alpha \longrightarrow (4)$

putting this value of β in (3),

$$2\{-1+2\alpha+1\} = \alpha+3$$

$$\Rightarrow 4\alpha = \alpha+3$$

$$\Rightarrow 3\alpha = 3$$

$$\Rightarrow \alpha = 1.$$

Putting $\alpha = 1$ in (4), we get

$$\beta = -1+2\cdot 1 = 1.$$

Hence, $I.F = x^{\alpha}y^{\beta} = xy$

Multiplying by xy, the given eqn. becomes

$$(xy^4 - 2y^2x^3)dx + (2x^2y^3 - x^4y)dy = 0$$

which is exact now.

$\therefore$ General soln. is

$$\int_{y\text{-const}} (xy^4 - 2y^2x^3)dx + \int (0)\,dy = 0$$

$$\Rightarrow \frac{1}{2}x^2y^4 - \frac{2}{4}x^4y^2 = c'$$

$$\Rightarrow x^2y^4 - x^4y^2 = c$$

$$\Rightarrow x^2y^2(y^2-x^2) = c \longleftarrow \text{Ans.}$$

α