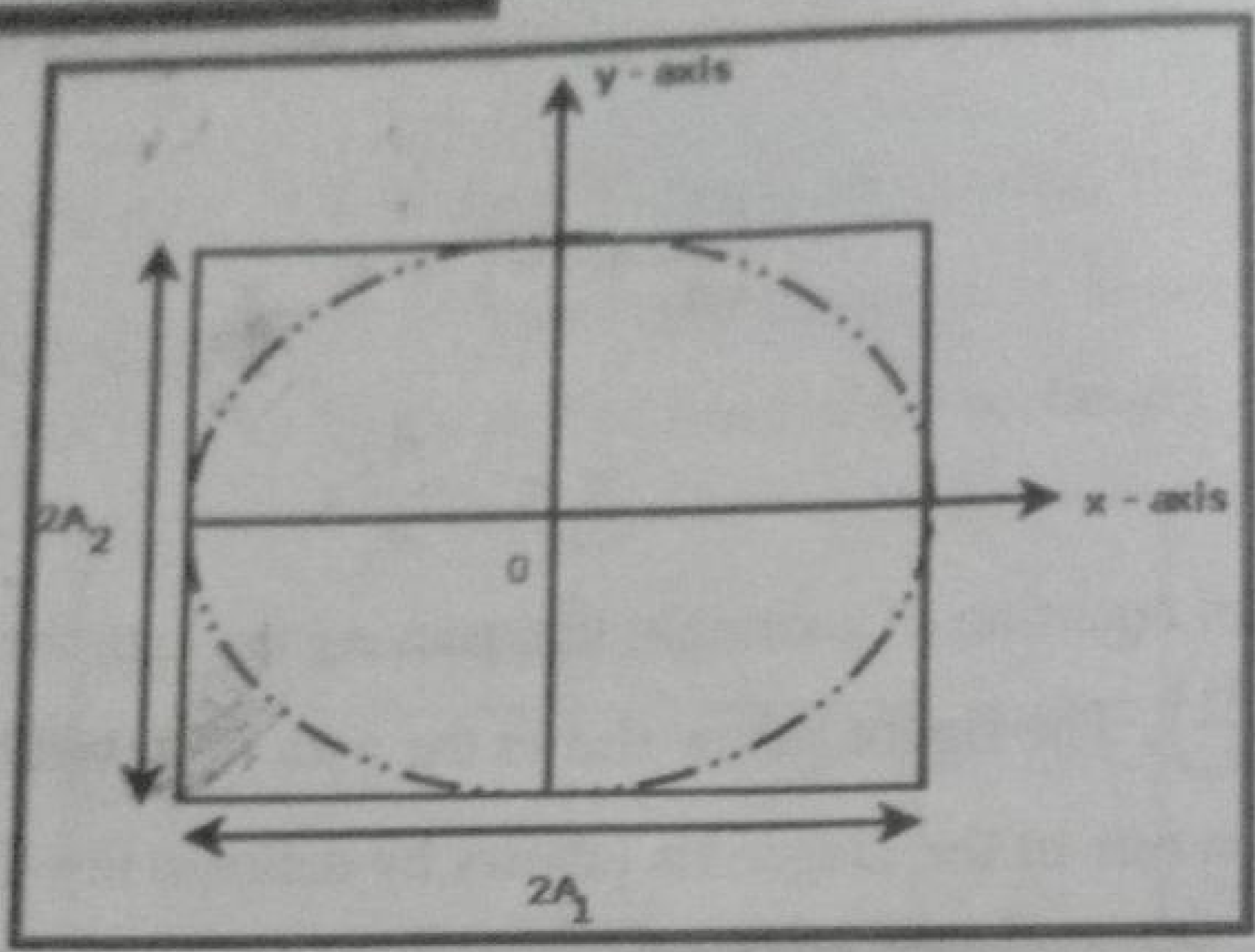

Figure 1.8: The elliptical path traced by the resultant motion of the particle when the plate difference, $\delta = \pi/2$

From equation (3), we get

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1 \qquad \dots(6)$$

The above equation is a standard equation of an ellipse with its axes along the x-axis and the y-axis and with its centre at the origin. The lengths of the major and the minor aces are $2A_1$ and $2A_2$, respectively.

The path traced by the particle(shown by the denoted line) is depicted in Fig 1.8.

In case the amplitudes of the two individual SHMs are equal $A_1 = A_2 = A$, i.e., the major and the minor axes are equal, then the ellipse reduces to a circle.

$$x^2 + y^2 = A^2 \qquad \dots(7)$$

Thus, the resultant motion of a particle due to superposition of two mutually perpendicular SHMs of equal amplitude and having a phase difference of $\pi/2$ is a circular motion. The circular motion may be clockwise or anticlockwise depending on which component leads the other.

1.8 Lissajous figures with equal and unequal frequency

In 1857 Lissajous demonstrated that when a particle is acted upon simultaneously two simple harmonic motions at right angles to each other, the resultant path traced out by the particle is a curve. The curves obtained by Lissajous after his name are called as Lissajous figures.

The nature of figures depends upon the following factors:
(i) amplitudes of the vibrations,
(ii) frequencies of the two vibrations, and
(iii) phase difference between the vibrations.

Here we shall discuss the analytical treatment of the Lissajous figures for different values of phase differences when frequencies of two rectangular vibrations are in the ratio 1:1 and 2:1 respectively.

Composition of two rectangular vibrations of different amplitudes but same frequency. Let us consider the case when two simple harmonic motions have the same frequency (or time period), one acting along the x-axis and the other along the y-axis. Let the two vibrations be represented by

$$x = a\sin(\omega t + \phi) \quad \ldots(1)$$

and

$$y = b\sin\omega t \quad \ldots(2)$$

when a and b are the amplitudes of x and y vibrations respectively. The x motion is ahead of the y motion by an angle ϕ i.e., the phase difference between two vibrations is ϕ.

The equation of resultant vibration can be obtained by eliminating t between equation (1) and (2).

From eq. (2), we have $\sin\omega t = \frac{y}{b}$

$$\therefore \quad \cos\omega t = \sqrt{1-\sin^2\omega t} = \sqrt{1-\left(\frac{y^2}{b^2}\right)}$$

Expanding eq.(1) and substituting the values of $\sin\omega$ and $\cos\omega t$, we get

$$\frac{x}{a} = \sin\omega t\cos\phi + \cos\omega t\sin\phi$$

or $$\frac{x}{a} = \frac{y}{b}\cos\phi + \sqrt{1-\frac{y^2}{b^2}}\sin\phi$$

or $$\frac{x}{a} - \frac{y}{b}\cos\phi = \sqrt{1-\frac{y^2}{b^2}}\sin\phi$$

Squaring both sides, we have

$$\left(\frac{x}{a} - \frac{y}{b}\cos\phi\right)^2 = \left(1-\frac{y^2}{b^2}\right)\sin^2\phi$$

or $\frac{x^2}{a^2}+\frac{y^2}{b^2}\cos^2\phi-\frac{2xy}{ab}\cos\phi=\sin^2\phi-\frac{y^2}{b^2}\sin^2\phi$

or $\frac{x^2}{a^2}+\frac{y^2}{b^2}(\cos^2\phi+\sin^2\phi)-\frac{2xy}{ab}\cos\phi=\sin^2\phi$

or $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos\phi=\sin^2\phi$ ... (3)

The equation represents an oblique ellipse, which is the resultant path of the particle. Here we consider the following important cases:

(i) When $\phi = 0$ (two vibrations are in phase). In this case

$\sin\phi = 0$ and $\cos\phi = 1$

The eq. (3) becomes

$$\frac{x^2}{y^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}=0$$

or $\left(\frac{x}{y}-\frac{y}{b}\right)^2=0$ or $\pm\left(\frac{x}{a}-\frac{y}{b}\right)=0$

or $\pm y=\pm\frac{b}{a}x$ ... (4)

This represents two coincident straight lines passing through the origin and inclines to x-axis at the angle θ, given by

$$\theta=\sin^{-1}\left(\frac{b}{a}\right)$$

This is the resultant path of the particle as shown in Fig 1.9(a)

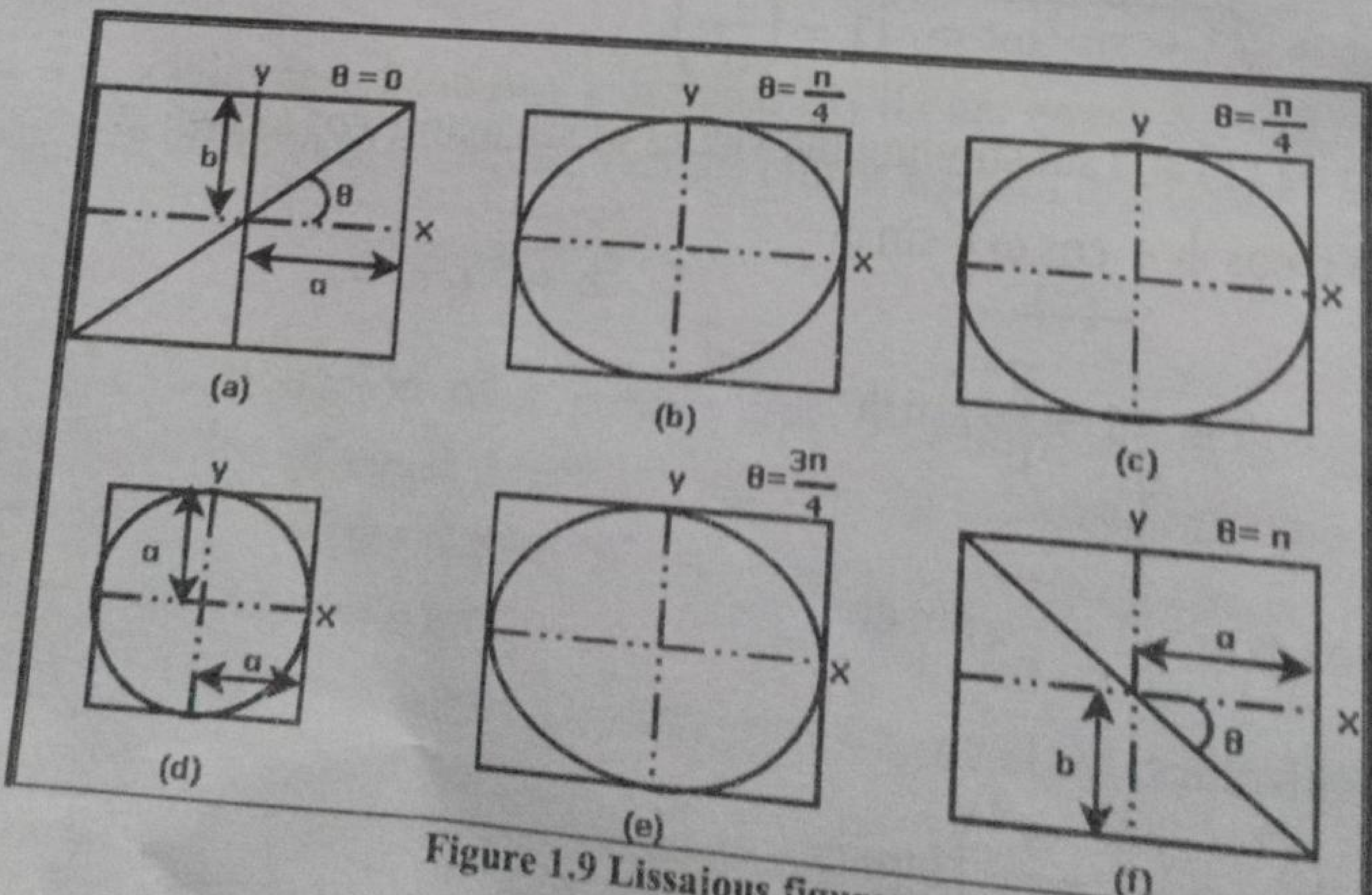

Figure 1.9 Lissajous figures

(ii) When $\phi = \pi/4$, we have

$\sin\phi = \frac{1}{\sqrt{2}}$ and $\cos\phi = \frac{1}{\sqrt{2}}$

Now eq. (3) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2} \qquad \dots(5)$$

This represents an oblique ellipse as shows in Fig. 1.9(b).

(iii) When $\phi = \pi/4$, we have

$\sin\phi = 1$ and $\cos\phi = 0$

The Eq. (3) reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \dots(6)$$

The resultant path is an ellipse whose major and minor axis coincide with the coordinate axes as shown in Fig 1.9(c). If a = b, then $x^2 + y^2 = a^2$. So the resultant path of the particle is a circle of radius a as shown in Fig 1.9(d).

(iv)When $\phi = 3\pi/4$, we have

$\sin\phi = \frac{1}{\sqrt{2}}$ and $\cos\phi = -\frac{1}{\sqrt{2}}$

The Eq. (3) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{2} \qquad \dots(7)$$

This represents an oblique ellipse as shown in Fig 1.9(e)

(v) When $\phi = \pi/4$, we have

$\sin\phi = 0$ and $\cos\phi = -1$

Now eq. (3) reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} = 0$$

or $\left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0$ or $\pm\left(\frac{x}{a} + \frac{y}{b}\right) = 0$

or $\pm y = \pm\frac{b}{a}x \qquad \dots(8)$

This again represents a pair of coincident straight lines passing through the origin and inclined to x axis at an angle θ, given by

$$\theta = \tan^{-1}\left(-\frac{b}{a}\right)$$

This is shown in Fig 1.9(f)

1.8.1 Composition of two rectangular vibrations of different amplitudes and of frequencies in the ratio 2:1.

Let us consider two simple harmonic vibrations having frequencies in the ratio 2:1 one along the X -axis and the other along the Y -axis. These vibrations are represented by

$$x = a\ \sin(2\omega t + \phi) \qquad \text{... (1)}$$

and
$$y = b\ \sin\omega t \qquad \text{... (2)}$$

Where a and b are their respective amplitudes and ϕ is the phase angle by which the x vibration is initially ahead of y-vibration. The equation of resultant vibration can be obtained by eliminated t be between equation (1) and (2).

From Eq. (2),

$$\sin\omega t = \frac{y}{b}, \therefore \cos\omega t = \sqrt{1 - \frac{y^2}{b^2}}$$

Expanding eq. (1), we get

$$\frac{x}{a} = \sin 2\,\omega t\ \cos\phi + \cos 2\,\omega t \sin\phi$$

$$= 2\sin 2\omega t\ \cos\omega\, t\ \cos\phi + (1 - 2\sin^2\omega t)\sin\phi$$

Substituting the values of sin ωt and cos ωt, we have

$$\frac{x}{a} = \frac{2y}{b}\sqrt{\left(1 - \frac{y^2}{b^2}\right)}\ \cos\phi + \sqrt{\left(1 - \frac{y^2}{b^2}\right)}\ \sin\phi$$

$$\text{or} \quad \frac{x}{a} - \left(1 - \frac{2y^2}{b^2}\right)\sin\phi = \frac{2y}{b^2}\sqrt{\left(1 - \frac{y^2}{b^2}\right)}\ \cos\phi$$

Squaring on both sides, we get

$$\frac{x^2}{a^2} + 1 - \left(\frac{2y^2}{b^2}\right)^2 \sin^2\phi - \frac{2x}{a}\left(1 - \frac{2y^2}{b^2}\right)\sin\phi$$

$$= \frac{4y^2}{b^2}\left(1 - \frac{y^2}{b^2}\right)\cos^2\phi$$

$$\text{or} \ \frac{x^2}{a^2} + \sin^2\phi + \frac{4y^4}{b^4}\sin^2\phi - \frac{4y^2}{b^2}\sin^2\phi - \frac{2x}{a}\sin^2\phi + \frac{4xy^2}{ab^2}\sin\phi$$

$$= \frac{4y^2}{b^2}(\sin^2\phi + \cos^2\phi) - \frac{4y^2}{b^2}(\sin^2\phi + \cos^2\phi) + \frac{4xy^2}{ab^2}\sin\phi = 0$$

$$\text{or} \ \left(\frac{x}{a} - \sin\phi\right)^2 + \frac{4y^2}{b^4} - \frac{4y^2}{b^2} + \frac{4xy^2}{ab^2}\sin\phi = 0$$

or $\left(\frac{x}{a}-\sin\phi\right)^2+\frac{4y^2}{b^2}\left(\frac{4y^2}{b^2}+\frac{x}{a}\sin\phi-1\right)=0$ ...(3)

This is the equation of a curve having two loops, which is the resultant path. Have we shall consider the following special cases:

(i) When $\phi = 0, \pi$ or 2π i.e., when the two component vibration are in phase. Substituting $\phi = 0$, in eq. (3), we have

$$\left(\frac{x}{a}\right)^2+\frac{4y^2}{b^2}\left(\frac{y^2}{b^2}-1\right)=0 \quad \text{...(4)}$$

This represents a figure of 8 as shown in Fig 1.10(a)

(ii) When $\phi = \pi/4$. In this case $\sin\phi = 1/\sqrt{2}$. Eq. (3) is

$$\left(\frac{x}{a}-\frac{1}{\sqrt{2}}\right)^2+\frac{4y^2}{b^2}\left(\frac{y^2}{b^2}-1+\frac{x}{a\sqrt{2}}\right)=0 \quad \text{...(5)}$$

This represents a figure of 8 as shown in fig. 1.10(b)

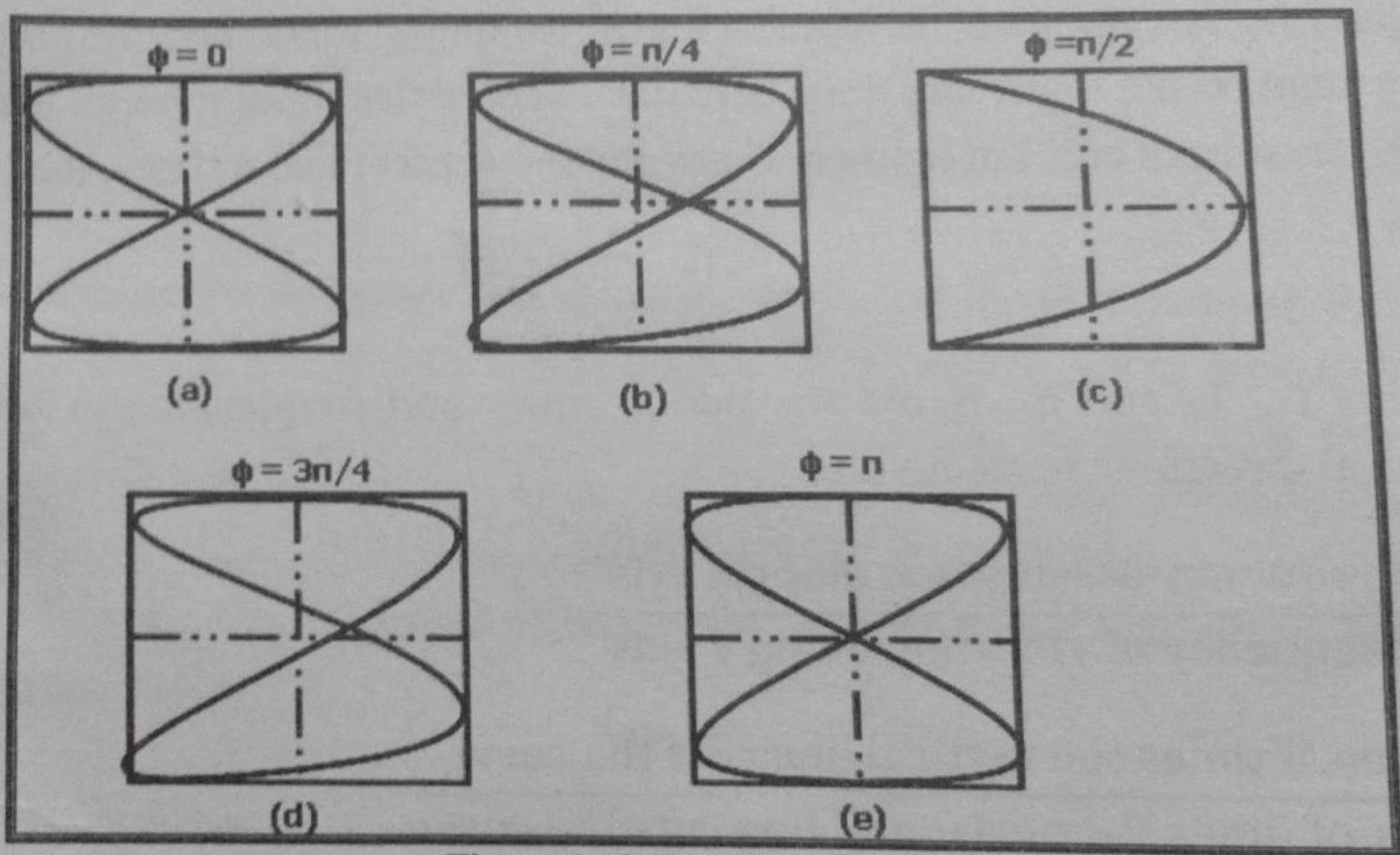

Figure 1.10: Lissajous figures

(iii) When $\phi = \pi/2$, we have $\sin\phi = 1$. Eq. (3) gives

$$\left(\frac{x}{a}-1\right)^2+\frac{4y^2}{b^2}\left(\frac{y^2}{b^2}+\frac{x}{a}-1\right)=0$$

or $\left(\frac{x}{a}-1\right)+\frac{4y^4}{b^4}+\frac{4y^2}{b^2}\left(\frac{x}{a}-1\right)=0$

or $\left[\left(\frac{x}{a}-1\right)+\frac{2y^2}{b^2}\right]^2=0$

This represents two coincident parabolas, the equation of each parabola being

$$\left(\frac{x}{a}-1\right)+\frac{2y^2}{b^2}=0 \text{ or } \frac{2y^2}{b^2}=-\left(\frac{x}{a}-1\right)$$

$$y^2=-\frac{b^2}{2a}(x-a) \quad \text{... (6)}$$

The pair of coincident parabolas symmetrical about x axis is shown in Fig. 1.10(c)

(iv) When $\phi = 3\pi/4$. In this $\sin\phi = 1/\sqrt{2}$. Eq. (3) again reduces to the same form as in case (ii). Hence the path of resultant vibration is the same.

(v) When $\phi = \pi$. In this case $\sin\phi = 0$. Hence the figure is gain obtained as shown in figure 1.10(e).

1.9 Uses of Lissajous Figures

(A) To compare the frequencies. If the ratio of two frequencies is a whole number, i.e. $n_1/n_2 = 1,2,3$, etc. then a steady figure will be traced. With the help of his steady figure, we can compare the frequencies of two tuning forks. For this purpose, we take any point on the figure and draw horizontal and vertical lines through that point. If the horizontal lines cuts the figure m times and the vertical line n times, then

$$\frac{T_1}{T_2}=\frac{n_2}{n_1}=\frac{m}{n}$$

where T_1, T_2 and n_1, n_2 are the period times and frequencies in horizontal and vertical directions respectively.

$$\therefore \frac{\text{Frequency of vibration along x axis}}{\text{Frequency of vibration along y axis}}$$

$$= \frac{\text{no. if times the vertical line cuts the curve}}{\text{no. of times the horizontal line cuts the curve}}$$

For example, let us consider the following figures shown in Fig.1.11

In Fig 1.11 (a), the horizontal line as well as vertical line cut the curve two times, hence the frequency ration is given by

$$\frac{n_1}{n_2}=\frac{n}{m}=\frac{2}{2}=\frac{1}{1}$$

Similarly for Fig 1.11 (b)

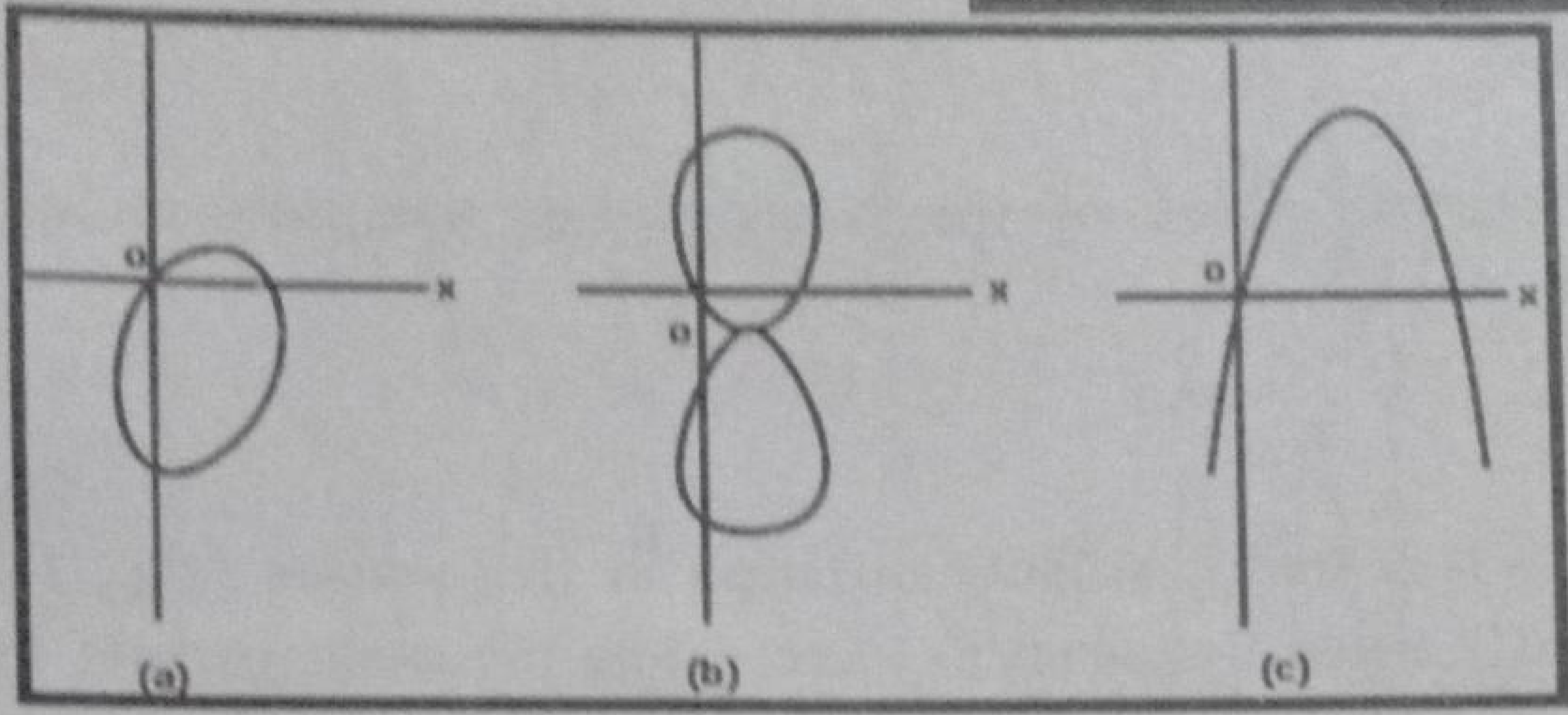

Figure 1.11

$$\frac{n_1}{n_2} = \frac{4}{2} = \frac{2}{1}$$

and for Fig 1.11(c)

$$\frac{n_1}{n_2} = \frac{1}{2}$$

(B) To determine the frequency of a tuning fork. Let the given tuning fork have unknown frequency n'. We take a tuning fork of nearly same frequency and adjusted the two tuning forks to vibrate in perpendicular planes. Now we obtain Lissajous figure. The Lissajous figure changes in shape continuously due to a slight difference in frequencies of two forks (the phase difference between two changes with time). Let the complete cycle change in t seconds then the difference in frequencies is given by

$$n' \sim n = \frac{1}{t} \text{ or } n' = n + \frac{1}{t}$$

Now we attach a little wax to the tuning fork of frequency n'. The waxing decreases the frequency of the tuning fork. The same experiment is again repeated. Let the time taken to complete on cycle of changes in this case be t_1. If t_1 is lesser than t, the frequency n' is given by

$$n'\left(n + \frac{1}{t}\right)$$

and if t_1 is greater than t_1, the frequency n' is given by

$$n' = \left(n - \frac{1}{t}\right)$$

Hence the correct value of the frequency of unknown tuning fork can be determined.

1.20 Beats

When two waves trains, slightly differing in frequencies (e.g., from two tuning forks of nearly equal frequencies) travel along the same straight line in the same direction, then the resultant amplitude is alternately maximum and minimum. Th the intensity of sound, which is proportional to the square of amplitude rises a falls (technically known as waxing and waning of sound) alternately with tim This phenomenon of waxing and waning of sound is called beats. The number waxing and waning in one second is called the frequency of beasts. This frequen of beats is equal to the difference in the frequencies of the sound waves.

Production of beats

The phenomenon of beats occurs as a result of interference between two sou waves of slightly different frequencies travelling along the same straight line in t same straight line in the same direction. Consider that a particular instant t_1 (F 1.12), the two waves meet in the same phase at a particular point. They reinfor to produce maximum sound intensity. After this instant, they get further and furth out of phase as their frequencies are slightly different. After a short time (at tim t_2) the two waves arrive at the point in the opposite phase. This happens when o wave gains half a vibration on the other. Now they produce minimum sou intensity. Again after some time i.e., at instant t_3 one wave gains one full vibratio on the other and the two waves are again in phase and produce maximum and o minimum constitute on beat. The number of beats per sec. is equal to the differen in frequencies of the sources.

Now we shall explain the production of beats by considering the case of two tunin forks of frequencies 256 and 254. Let the two forks start vibrating together in t same phase. After 1/4 second, the first fork completes its 64 vibrations while t second one has completed its $63_{1/2}$ vibrations. The two waves are now in opposi phase and produce minimum intensity. After 1/2 second, the two waves are aga in phase (phase difference is equal to λ) and produce maximum intensity. After 3 second, the first