Rules for finding Integrating Factor

Rule I: If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, a function of x only,

then $e^{\int f(x)dx}$ is an I.F.

Rule II: If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$, a fn. of y only,

then $e^{\int \{-g(y)\} dy}$ is an I.F.

Rule III: If $M\,dx + N\,dy = 0$ is homogeneous

and $Mx + Ny \neq 0$, then $\dfrac{1}{Mx+Ny}$ is an I.F.

Rule-IV: If the eqn. can be written in the form

$$y\,f(xy)\,dx + x\,g(xy)\,dy = 0,$$ where, $f(xy) \neq g(xy)$,

then $\dfrac{1}{xy[f(xy) - g(xy)]} = \dfrac{1}{Mx - Ny}$ is an I.F.

Rule-5: Let the eqn. be of the form

$$x^a y^b (my\,dx + nx\,dx) + x^c y^d (\mu y\,dx + \nu x\,dy),$$

where $a, b, c, d, m, n, \mu, \nu$ are consts. Then an I.F is $x^\alpha y^\beta$, where α and β are so chosen that after multiplying by $x^\alpha y^\beta$ the eqn. becomes exact.

Examples of Rule-I

Ex. 1. Solve

$$(x^2+y^2+x)\,dx + xy\,dy = 0$$

Soln Here,

$$M = x^2+y^2+x$$

$$N = xy.$$

$$\therefore \frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = y. \quad \therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

$\therefore$ The given eqn is not exact.

Now we see that

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-y}{xy} = \frac{1}{x}, \text{ a fn of } x \text{ alone.}$$

Hence

$$\text{I.F.} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

Now, multiplying by I.F. i.e. by x, The given eqn. becomes

$$(x^3+xy^2+x^2)\,dx + x^2y\,dy = 0$$

which is exact now.

Here,
$M = x^3+xy^2+x^2$
$N = x^2y$
$\therefore \frac{\partial M}{\partial y} = 2xy$
$\frac{\partial N}{\partial x} = 2xy$

$\therefore$ The G.S. is

$$\int_{y\text{-constant}} (x^3+xy^2+x^2)\,dx + \int (0)\cdot dy = c'$$

$$\Rightarrow \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} + 0 = c'$$

$$\Rightarrow 3x^4 + 6x^2y^2 + 4x^3 = C, \quad C = 12c'$$

Which is the reqd. solⁿ.

=x=

Example of rule-II :

Ex. Solve

$$(3x^2y^4+2xy)dx+(2x^3y^3-x^2)dy=0$$

Sol: Here,

$$M=3x^2y^4+2xy, \quad N=2x^2y^3-x^2.$$

$$\therefore \frac{\partial M}{\partial y}=12x^2y^3+2x, \quad \frac{\partial N}{\partial x}=6x^2y^3-2x$$

$$\therefore \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$ & so the given eqn. is not exact.

Now, we see that

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=\frac{12x^2y^3+2x-6x^2y^3+2x}{3x^2y^4+2xy}$$

$$=\frac{6x^2y^3+4x}{y\{3x^2y^3+2x\}}=\frac{2}{y}$$

which is a function of y alone.

$$\therefore I.F.=e^{\int(-\frac{2}{y})dy}=e^{-2\log y}=e^{\log(y^{-2})}=\frac{1}{y^2}$$

$\therefore$ Multiplying the given eqn by $\frac{1}{y^2}$, we get

$$\left(3x^2y^2+\frac{2x}{y}\right)dx+\left(2x^3y-\frac{x^2}{y^2}\right)dy=0$$

which is exact now.

In this case,
$$\frac{\partial M}{\partial y}=6x^2y-\frac{2x}{y^2}=\frac{\partial N}{\partial x}$$

$\therefore$ G. Soln. is

$$\int_{y\text{-const.}}\left(3x^2y^2+\frac{2x}{y}\right)dx+\int(0)dy=c'$$

$$\Rightarrow x^3y^2+\frac{x^2}{y}+0=c'$$

$$\Rightarrow x^3y^3+x^2=cy \longleftarrow \text{Ans.}$$