The advantage with the vector method is that it can easily be extended to more than two vectors. For example, if we have 3 vectors,

$$x_1 = A_1 \sin \omega t$$

$$x_2 = A_2 \sin(\omega t + \delta_1)$$

$$x_3 = A_3 \sin(\omega t + \delta_2)$$

They can be represented as shown in the figure below

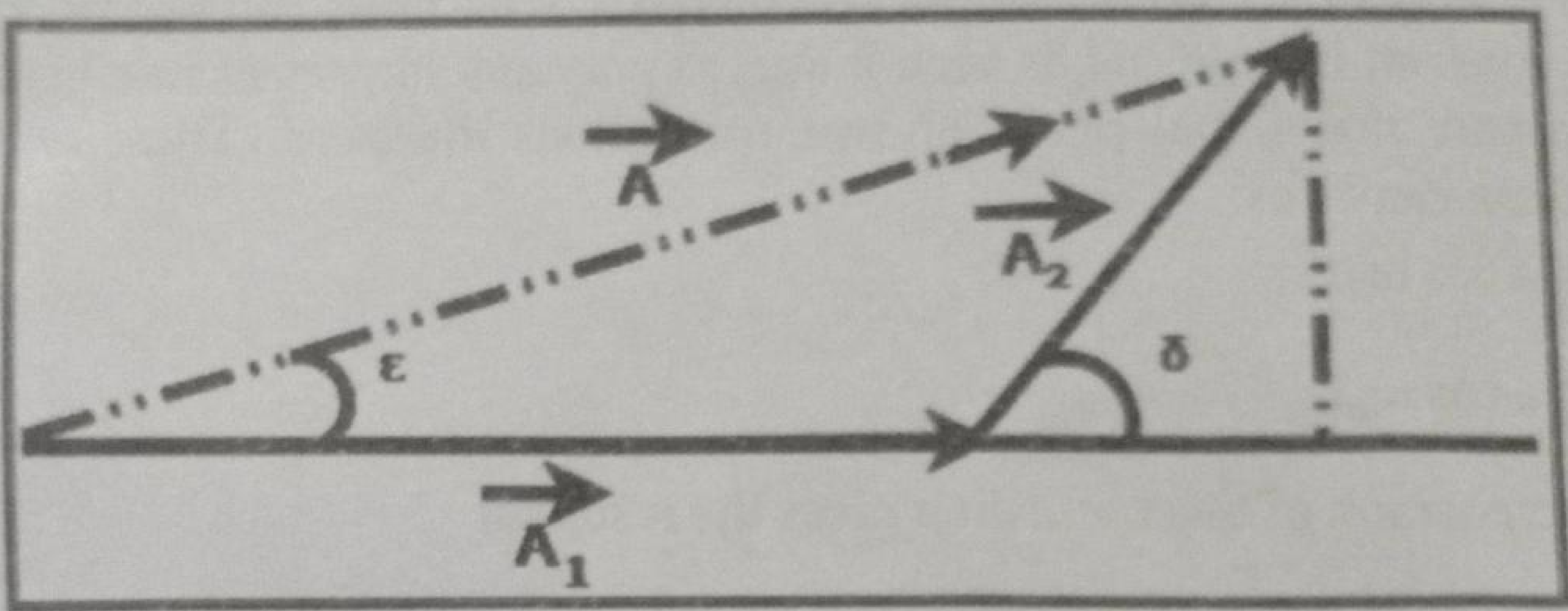

Figure 1.2: There vector representing SHMs along with the resultant vector

The resultant vector is given by

$$x = A \sin(\omega t + \varepsilon)$$

Using trigonometry, we can see that

$$A = \sqrt{(A_1 + A_2 \cos \delta_1 + A_3 \cos \delta_2)^2 + (A_2 \sin \delta_1 + A_3 \sin \delta_2)^2}$$

And

$$\tan \varepsilon = \frac{Height}{Base} = \frac{A_2 \sin \delta_1 + A_3 \sin \delta_2}{A_1 + A_2 \cos \delta_1 + A_3 \cos \delta_2}$$

1.5 Superposition of two collinear harmonic osciilations of different frequencies

In numerous cases, we have to deal with superposition of two or more harmonic oscillations having different angular frequencies. The microphone diaphragm and human eardrums are simultaneously subjected to different vibrations. For ease, we shall first consider superposition of two harmonic oscillations having same amplitude a but somewhat different frequencies ω_1 and ω_2 such that $\omega_1 > \omega_2$:

$x_1 = a\cos(\omega_1 t + \Phi_1)$

$x_2 = a\cos(\omega_2 t + \Phi_2)$

Phase difference between the two harmonic vibrations is:

$$\Phi = (\omega_1 - \omega_2)t + (\Phi_1 - \Phi_2)$$

The first term $(\omega_1 - \omega_2)t$ changes continuously with time. But second term $(\Phi_1 - \Phi_2)$ is constant in time and as such it doesn't play any important role here. Thus, we may assume that initial phase of two oscillations are zero. Then, two harmonic oscillations can be as:

$x_1(t) = a\cos\omega_1 t$

$x_2(t) = a\cos\omega_2 t$

The superposition of two oscillation gives the resultant

$$x(t) = x_1(t) + x_2(t) = a\cos(\omega_1 t + \omega_2 t)$$

This equation cab be rewritten as

$$x(t) = 2a\cos\left(\frac{(\omega_1 - \omega_2)}{2}\right)t\cos\left(\frac{(\omega_1 + \omega_2)}{2}\right)t$$

This is the oscillatory motion with angular frequency

$\left(\frac{(\omega_1 - \omega_2)}{2}\right)$ and amplitude 2acos $\left(\frac{(\omega_1 - \omega_2)}{2}\right)t$

The average angular frequency

$\omega_{av} = \left(\frac{(\omega_1 - \omega_2)}{2}\right)$ and the modulated angular frequency

$$\omega mod = \left(\frac{(\omega_1 - \omega_2)}{2}\right)$$

We find that amplitude $a_{mod}(t) = 2\cos\omega mod t$differs with the frequency

$$\frac{\omega_{mod}}{2\pi} = \frac{(\omega_1 - \omega_2)}{4\pi}$$

This also signifies that in one complete cycle modulated amplitude takes values of 2a, 0, -2a, 0 and 2a for $\omega_{mod}\, t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π, respectively. Resultant oscillation can be written as

$$x(t) = a_{mod}(t) \cos \omega_{av} t$$

This equation looks like the equation of SHM. But this resemblance is misleading. Modulated amplitude and phase constant are respectively provided by:

$$a_{mod}(t) = [a_1{}^2 + a_2{}^2 + 2a_1a_2 \cos(2\omega mod t)]^{1/2}$$

And $\theta_{mod} = [(a_1 - a_2) \sin \omega_{mod}\, t/(a_1 + a_2 \cos \omega_{mod}\, t)]$

1.6 Superposition of many harmonic oscillations of same frequency (method of vector addition)

This method is based on fact that displacement of the harmonic oscillation is projection of the uniform circular motion on diameter of circle. Thus, it is significant to understand connection between SHM and uniform circular motion.

Uniform Circular Motion and SHM

Let us assume that a particle moves in the circle with constant speed V. Radius vector joining centre of circle and position of particle on the circumference will rotate with the constant angular frequency. We take x-axis to be along the direction of radius vector at time t = 0. Then angle made by the radius vector with x-axis at any time t will be given by

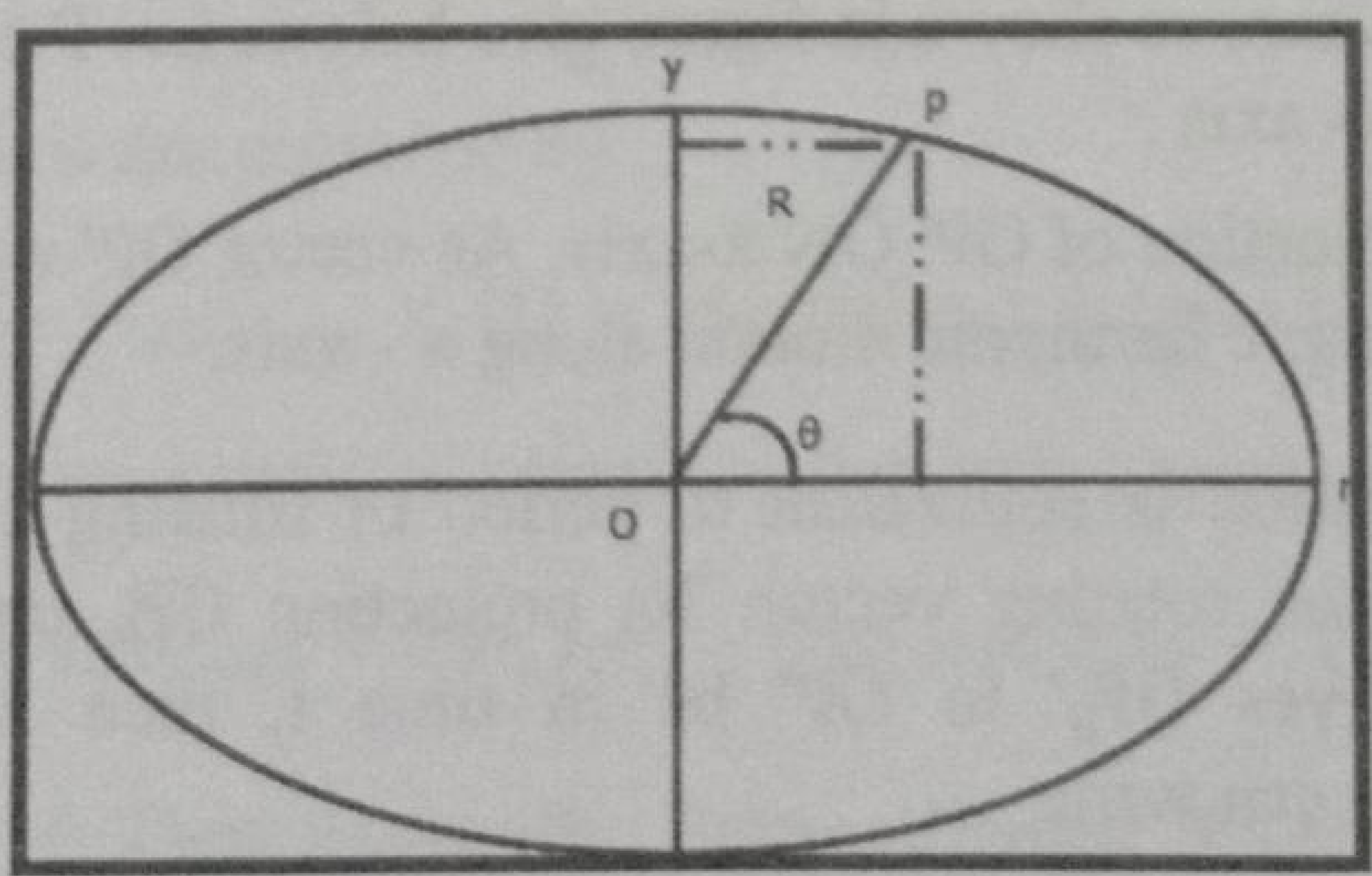

Figure 1.3: uniform circular motion

$\theta = \text{Length of} \frac{\text{arc}}{\text{radiius}} \text{of the circle} = \frac{Vt}{R}$

The x- and y - components of position of particle at time t are

$x = R\cos\theta$ and $y = R\sin\theta$

Therefore, $\frac{dx}{dt} = -R\sin\theta\frac{d\theta}{dt}$

$= -\omega_0 R\sin\theta$

As $\frac{d\theta}{dt} = \omega_0 = \frac{V}{R}$

Similarly, you can write

$\frac{dy}{dt} = \omega_0 R\ \cos\theta$

Differentiating again with respect to time, we get

$$\frac{d_2x}{dt_2} = -\omega_0{}^2x \text{ and } \frac{d^2y}{dt^2} = -\omega_0 2y$$

These expressions show that when the particle moves uniformly in the circle, its projections along x- and y- axes execute SHM. A simple harmonic motion may be viewed as the projection of uniformly rotating vector on reference axis.

Assume that vector OP' with |OP'| = a_0 is rotating with angular frequency ω_0 in anticlockwise direction, as shown in figure give below. Let P be foot of perpendicular drawn from P' on x - axis .

Then OP = x is projection of OP' ON x-axis. As vector OP' rotates at constant speed, point P executes simple harmonic motion along x - axis.

Its period of oscillation is equivalent to period of rotating vector OP' Let OP_0 be initial position of the rotating vector. Its projection OP_0 on x-axis is $a_0 \cos\Phi$. If rotating vector moves OP_0' to OP' be in time t, then $\angle P'OP_0'\omega_0 t$ and $\angle P'_0 = (\omega_0 t + \Phi)$ Then we can write

$OP = OP'\cos\angle P'O^X$

Or $x = a_0\cos(\omega_0 t + \Phi)$

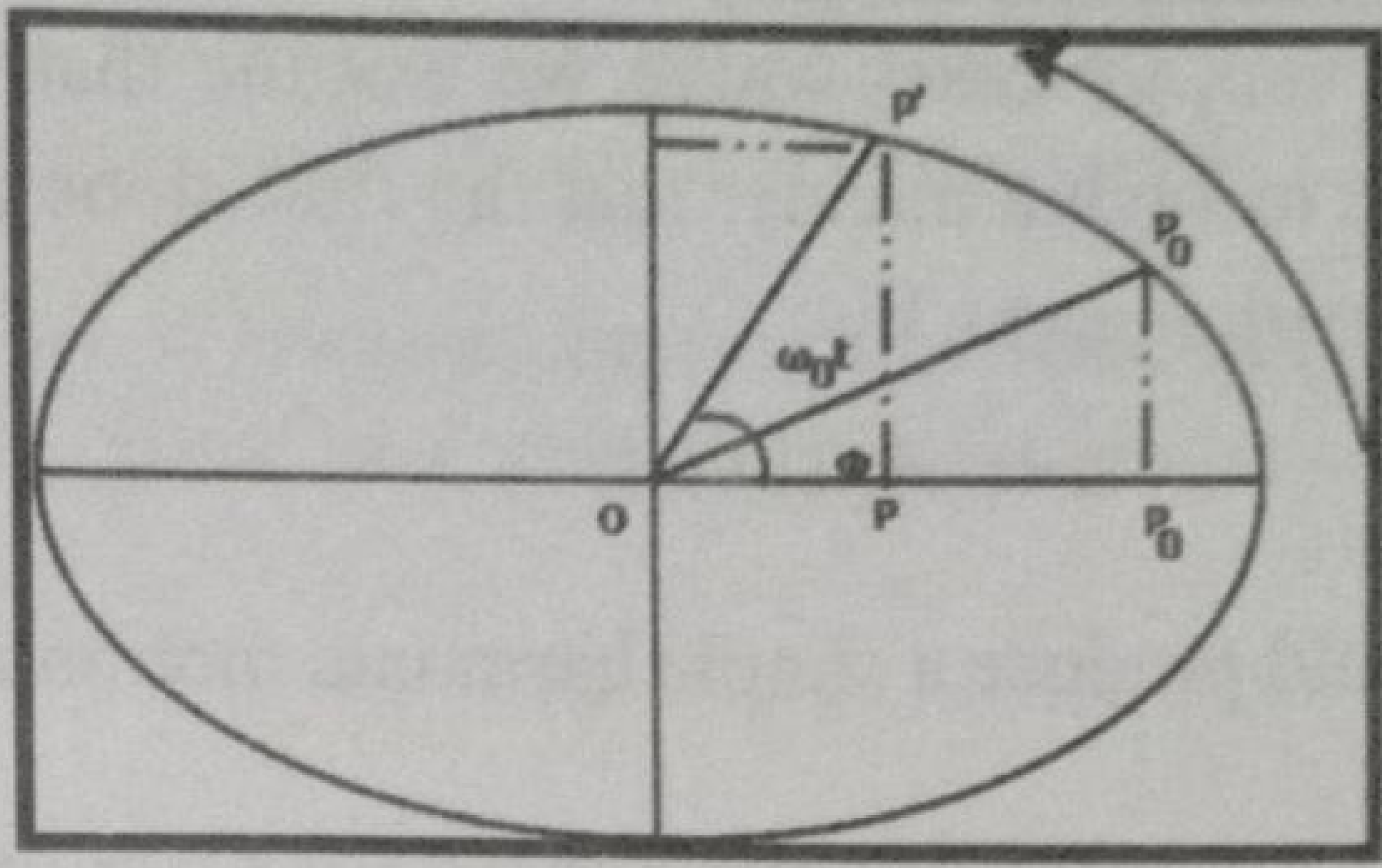

Figure 1.4

Therefore, point P executes simple harmonic motion along axis. If you project OP' on y-axis, you will find that point corresponding to the foot of the normal satisfies equation

$$y = a_0 \sin(\omega_0 t + \Phi)$$

This signifies that rotating vector can, in general, be resolved in two orthogonal components, and we can write

$$r = x_x + y_y$$

When x_x and y_y are unit vectors along x-and y-axes, respectively.

Vector Addition:

Let us Now consider superposition of n harmonic oscillations, all having same amplitude a_0 and angular frequency ω_0. Initial phases of successive oscillations differ by Φ_0. Let first of these oscillations be defined by equation

$$x_1(t) = a_0 \cos \omega_0 t$$

Then, other oscillations are provided by:

$$x_2(t) = a_0 \cos(\omega_0 t + \Phi_0) \ldots\ldots x_n(t) = a_0 \cos[\omega_0 t + (n-1)\Phi_0]$$

From principle of superposition, resultant oscillation is expressed by

$$x(t) = a_0[\cos \omega_0 t + \cos(\omega_0 t + 2\Phi_0) + \cdots \cos(\omega_0 t + (n-1)\Phi_0)]$$

1.7 Superposition of Two Mutually Perpendicular Harmonic Oscillations of same Frequency

Let's now study a simpler case, where we assume that two independent forces are acting on a particle in such a manner that the first alone produces a simple harmonic motion in the x-direction given by

$$x = A_1 \sin \omega t \qquad \text{... (1)}$$

and the second would produce a simple harmonic motion in the y-direction given by

$$y = A_2 \sin(\omega t + \delta) \qquad \text{... (2)}$$

Thus, we are actually considering the superposition of two mutually perpendicular SHMs which have equal frequencies. The amplitudes may be different and their phases differ by δ. The resultant motion of the particle is a combination of the two SHMs.

The position of the particle at any time t is given by (x, y) where a and yare given by the above equations. The resultant motion is, thus, two-dimensional and the path of the particle is , in general, an ellipse. The equation of the path traced by the particle is obtained by eliminating t from equation (1) and (2).

From equation (1), we get

$$\sin \omega t = \frac{x}{A_1}; \text{which gives } \cos \omega t = \sqrt{1 - \left(\frac{x}{A_1}\right)^2}$$

Putting these values in equation (2), we get

$$y = A_2 \sin(\omega t + \delta) = A_2[\sin \omega t \cos \delta + \cos \omega t \sin \delta]$$

$$= A_2\left[\left(\frac{x}{A_1}\right)\cos \delta + \left(\sqrt{1 - \left(\frac{x}{A_1}\right)^2}\right)\right]$$

$$or, \quad \left(\frac{y}{A_2} - \frac{x}{A_1}\cos \delta\right)^2 = \left(1 - \left(\frac{x}{A_1}\right)^2\right)\sin^2 \delta$$

$$\therefore \quad \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy\cos \delta}{A_1 A_2} = \sin^2 \delta \qquad \text{... (3)}$$

As we can, equation (3) is an equation of ellipse. Thus, we may conclude that the resultant motion of the particle is along an elliptical path.

Equation (3) shows that x remains between -A1and A2 and that of y remains between -A2 and A2. Thus, the particle always remains inside the rectangle defined by

$x \pm A_1 and\ y = \pm A_2$

The ellipse given by equation (3) is shown in the figure 1.5:

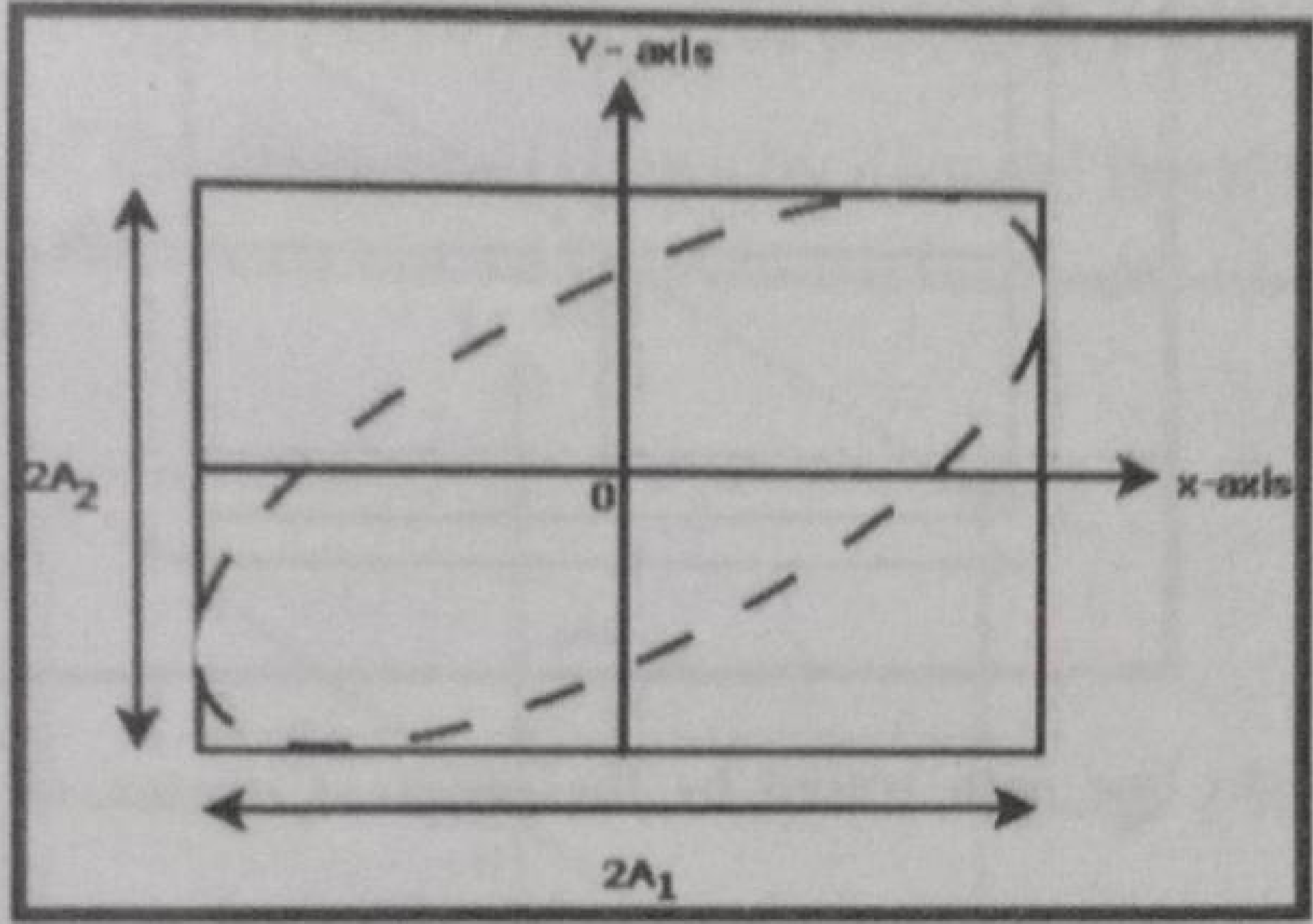

Figure 1.5: Elliptical path followed by a particle on which two independent SHMs, which are perpendicular to each other, act simultaneously

Special Cases

- The two component SHMs are in phase, $\delta = 0$
- The two component SHMs are out of phase, $\delta = \pi$
- The phase different between the two component SHMs, $\delta = \pi/2$

Let us now obtain the resultant motion of the particle under the special cases.

(a) When the two superposing SHMs are in phase, $\delta = 0$ and equation (3) reduces to

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy}{A_1 A_2} = 0$$

Or,

$$\left(\frac{y}{A_2} - \frac{x}{A_1}\right)^2 = 0$$

$$\therefore \quad y = \frac{A_2}{A_1} x \qquad ...(4)$$

Equation (4) is an equation of a straight line passing through the origin and having a slope of

Equation (4) is an equation of a straight line passing through the origin and having a slope of $\tan^{-1}\left(\frac{A_2}{A_1}\right)$. The figure below shows the particle in this case. The particle moves on the diagonal (shown by the dotted line) of the rectangle.

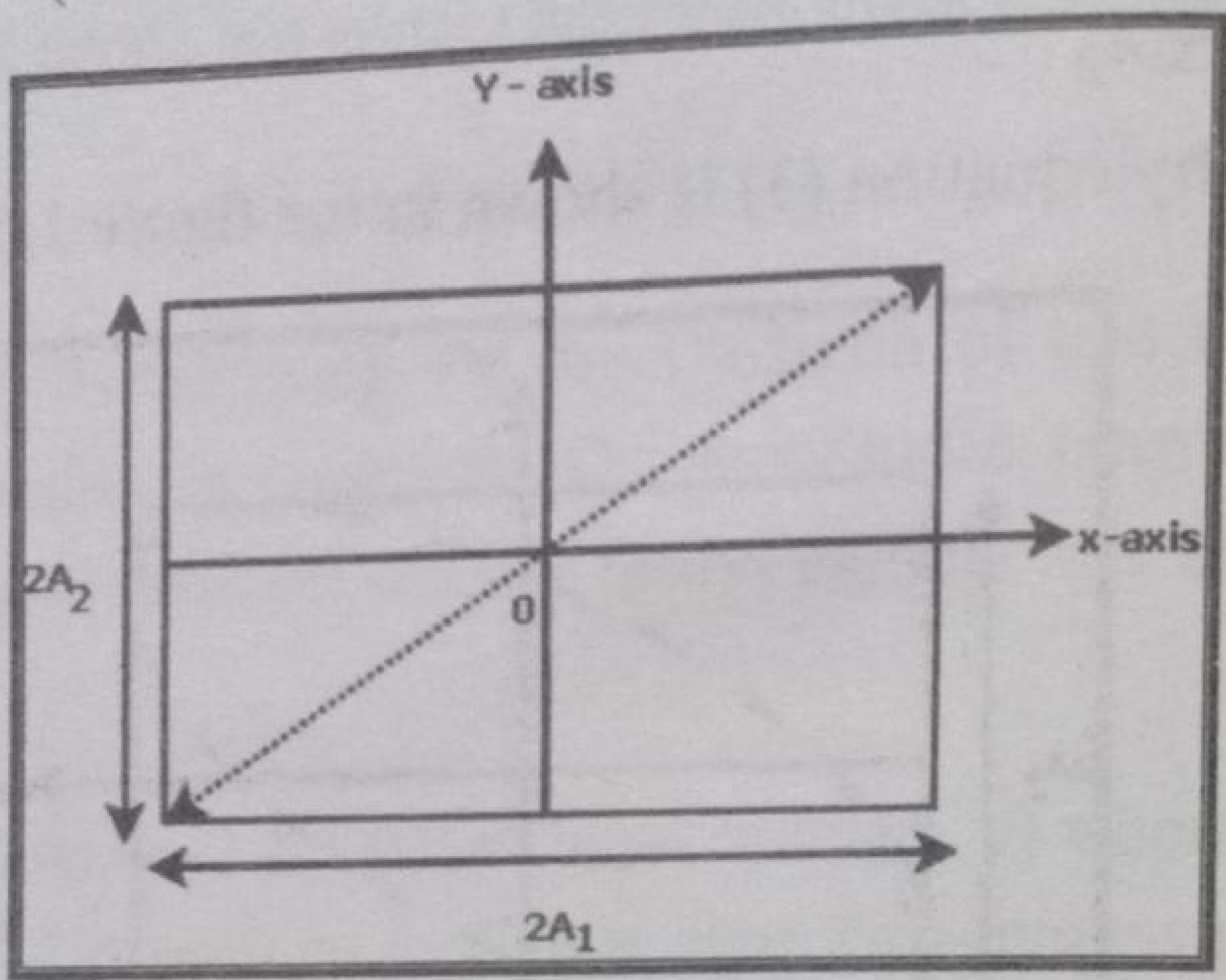

Figure 1.6: The straight line path traced by the resultant motion of the particle when the phase difference, δ = 0.

Equation (4) can also be obtained directly from equation (1) and (2) putting δ = 0. The displacement of the particle on this straight line at time is

$$r = \sqrt{x^2 + y^2}$$

$$= \sqrt{(A_1 \sin \omega t)^2 + (A_2 \sin \omega t)} = \sqrt{A_1^2 + A_2^2} \sin \omega t$$

Thus, we can see that the resultant motion is also SHM with the same frequency and phase as the component motions. However, the amplitude of the resultant SHM is $\sqrt{A_1^2 + A_2^2}$

(b) When the two superposing SHMs are t of phase, the phase difference between them is δ = π. Thus, from equation (3), we get

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} + \frac{2xy}{A_1 A_2} = 0$$

or,

$$\left(\frac{y}{A_2}+\frac{x}{A_1}\right)^2 = 0$$

∴

$$y = -\frac{A_2}{A_1}x \qquad ...(5)$$

Equation (5) is an equation of a straight line passing through the origin and having a slope $\tan^{-1}\left(-\frac{A_2}{A_1}\right)$. The figure below shows the path followed by the particles. The particle moves on one of the diagonals (shows by denoted line)f the rectangle.

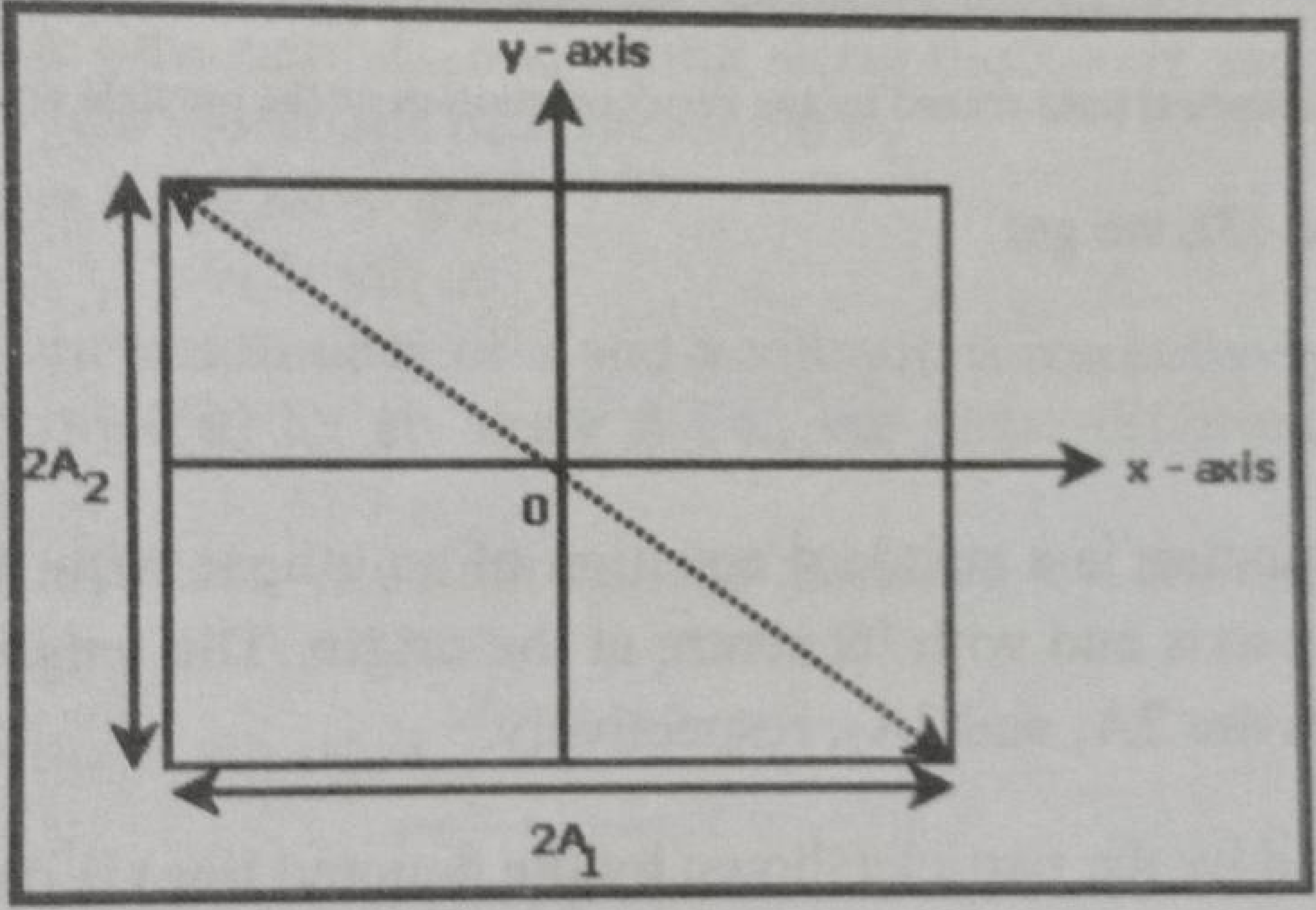

Figure 1.7 : The straight line path traced by the resultant motion of the particle when the phase difference, δ =π

Equation (5) can also be obtained directly on the basis of equation (1) and (2) and putting $\delta = \pi$. Further, the displacement of the particle on the particle on this straight line path at a given time is

$$r = \sqrt{x^2 + y^2} = \sqrt{(A_1 \sin \omega t)^2 + (A_2 \sin(\omega t + \pi))^2}$$

$$= \sqrt{(A_1 \sin \omega t)^2 + (-A_2 \sin \omega t)^2} = \sqrt{A_1^2 + A_2^2} \sin \omega t$$

Thus, we can see that the resultant motion is also SHM with the same frequency as the component motions. The amplitude of the resultant SHM is $\sqrt{A_1^2 + A_2^2}$.

(c) When the phase difference between the two component SHM is $\delta = \pi/2$.

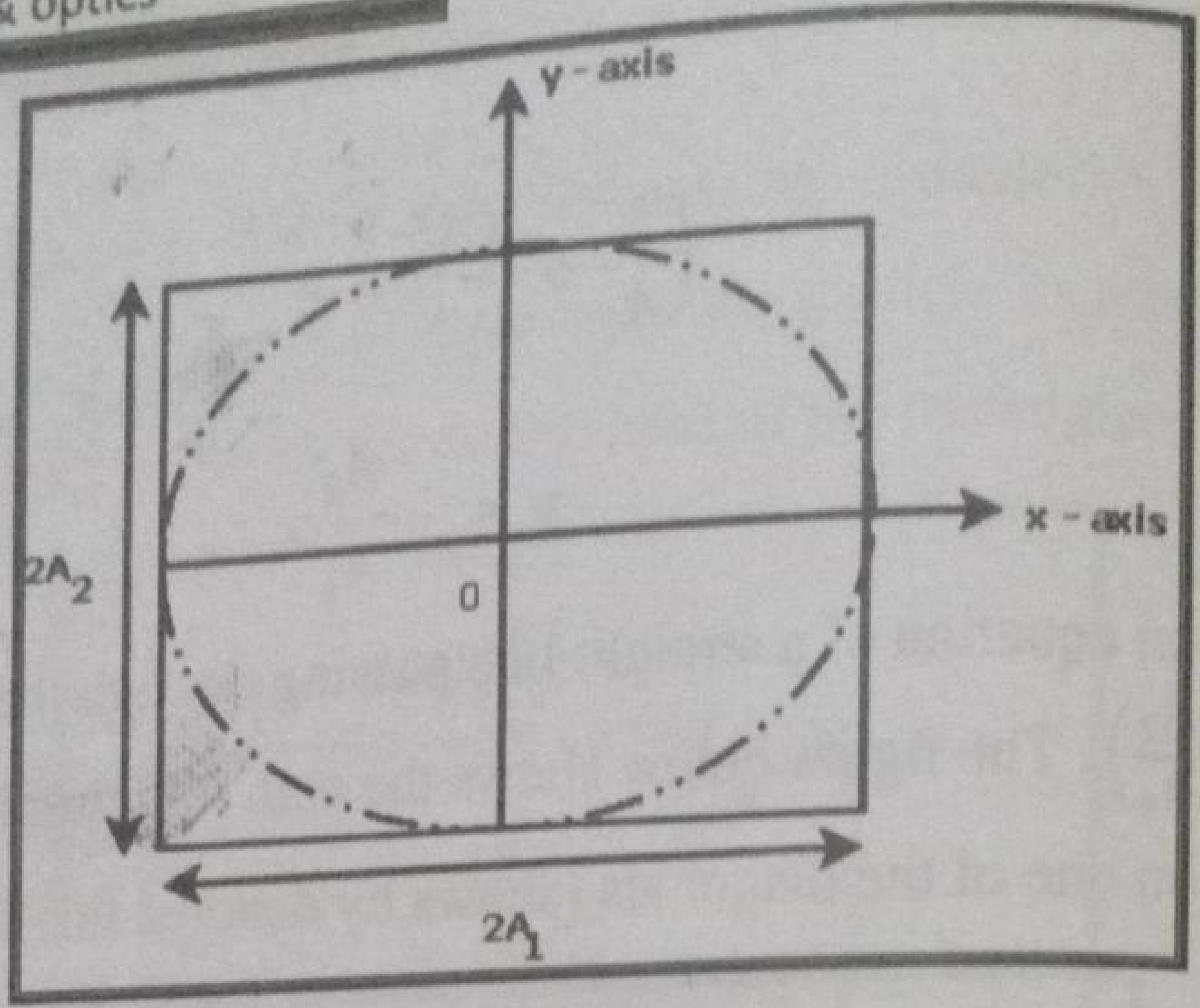

Figure 1.8: The elliptical path traced by the resultant motion of the particle when the plate difference, δ= π/2

From equation (3), we get

$$\frac{x^2}{A_1^2}+\frac{y^2}{A_2^2}=1 \qquad \dots(6)$$

The above equation is a standard equation of an ellipse with its axes along the x-axis and the y-axis and with its centre at the origin. The lengths of the major and the minor aces are $2A_1$ and $2A_2$, respectively.

The path traced by the particle(shown by the denoted line) is depicted in Fig 1.8.

In case the amplitudes of the two individual SHMs are equal $A_1 = A_2 = A$, i.e., the major and the minor axes are equal, then the ellipse reduces to a circle.

$$x^2+y^2=A^2 \qquad \dots(7)$$

Thus, the resultant motion of a particle due to superposition of two mutually perpendicular SHMs of equal amplitude and having a phase difference of $\pi/2$ is a circular motion. The circular motion may be clockwise or anticlockwise depending on which component leads the other.

1.8 Lissajous figures with equal and unequal frequency

In 1857 Lissajous demonstrated that when a particle is acted upon simultaneously