Integrating Factors: If an eqn. becomes exact after is has been multiplied by a function of x and y, then such a fn. is called an integrating factor.

Number of Integrating Factors:

∴ There are infinite number of integrating factors for an eqn

$$M\,dx + N\,dy = 0$$

Integrating factors by Inspection:

Sometimes an integrating factor can be found by inspection. For this we should study (or, remember) the following results:

① $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$

② $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

③ $d(xy) = x\,dy + y\,dx$

④ $d\left(\frac{y^2}{x}\right) = \frac{2xy\,dy - y^2\,dx}{x^2}$

⑤ $d\left(\frac{x^2}{y}\right) = \frac{2yx\,dx - x^2\,dy}{y^2}$

⑥ $d\left(\log\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{xy}$

⑦ $d\left[\log(xy)\right] = \frac{x\,dy + y\,dx}{xy}$

⑧ $d\left(\frac{e^x}{y}\right) = \frac{ye^x\,dx - e^x\,dy}{y^2}$

⑨ $d\left[\tan^{-1}\frac{y}{x}\right] = \frac{\frac{x\,dy - y\,dx}{x^2}}{1 + \left(\frac{y}{x}\right)^2} = \frac{x\,dy - y\,dx}{x^2 + y^2}$

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Solve By inspection

Eg. ① $(x+y^2)dy + (y-x^2)dx = 0$

Sol^n Given eqn. is

$$(x+y^2)dy + (y-x^2)dx = 0$$

$$\Rightarrow x\,dy + y^2\,dy + y\,dx - x^2\,dx = 0$$

$$\Rightarrow (x\,dy + y\,dx) + y^2dy - x^2dx = 0, \quad \text{(regrouping)}$$

$$\Rightarrow d(xy) + y^2dy - x^2dx = 0$$

Integrating,

$$\int d(xy) + \int y^2dy - \int x^2dx = c$$

$$\Rightarrow xy + \frac{y^3}{3} - \frac{x^3}{3} = c'$$

$$\Rightarrow y^3 - x^3 + 3xy = c \quad \leftarrow \text{Ans.}$$

Solve $y\,dx - x\,dy + 3x^2y^2e^{x^3}\,dx = 0$

Sol^n The eqn. can be written as

$$\frac{y\,dx - x\,dy}{y^2} + 3x^2e^{x^3}\,dx = 0$$

$$\Rightarrow d\left(\frac{x}{y}\right) + d\left(e^{x^3}\right) = 0$$

$$\Rightarrow \int d\left(\frac{x}{y}\right) + \int d\left(e^{x^3}\right) = c, \text{ integrating}$$

$$\Rightarrow \frac{x}{y} + e^{x^3} = c \quad \longleftarrow \text{Ans.}$$

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(3) Solve

$$x\,dy - y\,dx + 2x^3\,dx = 0$$

Sol^n: The given eqn. can be written as

$$\frac{x\,dy - y\,dx}{x^2} + 2x\,dx = 0$$

$$\Rightarrow d\left(\frac{y}{x}\right) + 2x\,dx = 0$$

$$\Rightarrow \int d\left(\frac{y}{x}\right) + 2\int x\,dx = C', \text{ integrating}$$

$$\Rightarrow \frac{y}{x} + x^2 = C'$$

$$\Rightarrow y + x^3 = Cx \leftarrow \text{Ans.}$$

Ex. $y(2xy + e^x)\,dx - e^x\,dy = 0$

Sol^n: From given eqn.

$$2xy^2\,dx + y e^x\,dx - e^x\,dy = 0$$

$$\Rightarrow 2x\,dx + \frac{e^x}{y}\,dx - \frac{e^x}{y^2}\,dy = 0$$

$$\Rightarrow d(x^2) + \frac{y e^x\,dx - e^x\,dy}{y} = 0$$

$$\Rightarrow d(x^2) + d\left(\frac{e^x}{y}\right) = 0$$

$$\Rightarrow \int d(x^2) + \int d\left(\frac{e^x}{y}\right) = C, \text{ integrating}$$

$$\Rightarrow x^2 + \frac{e^x}{y} = C \leftarrow \text{Ans.}$$

Note: → Here $\frac{1}{y^2}$ is multiplied the whole eqn. Hence $\frac{1}{y^2}$ is integrating factor which we have found by inspection. //
