$$
\begin{aligned}
f'(Z) &= \lim_{\Delta Z \to 0}\left[\frac{1}{\Delta Z}[f(Z+\Delta Z) - f(Z)]\right] \\
&= \lim_{\Delta Z \to 0}\left[\frac{1}{2\pi i\,\Delta Z}\left\{\oint_c \frac{f(\xi)\,d\xi}{\xi - Z - \Delta Z} - \oint_c \frac{f(\xi)\,d\xi}{\xi - Z}\right\}\right] \\
&= \frac{1}{2\pi i}\oint_c \frac{f(\xi)\,d\xi}{(\xi - Z)^2}.
\end{aligned}
$$

For the nth derivative of $f(Z)$ with respect to Z, we have

$$
f^{(n)}(Z_0) \equiv f^{(n)}(Z)\big|_{Z=Z_0} = \frac{n!}{2\pi i}\oint_c \frac{f(\xi)\,d\xi}{(\xi - Z_0)^{n+1}} \tag{3.42}
$$

when $f(\xi)$ is analytic within C.

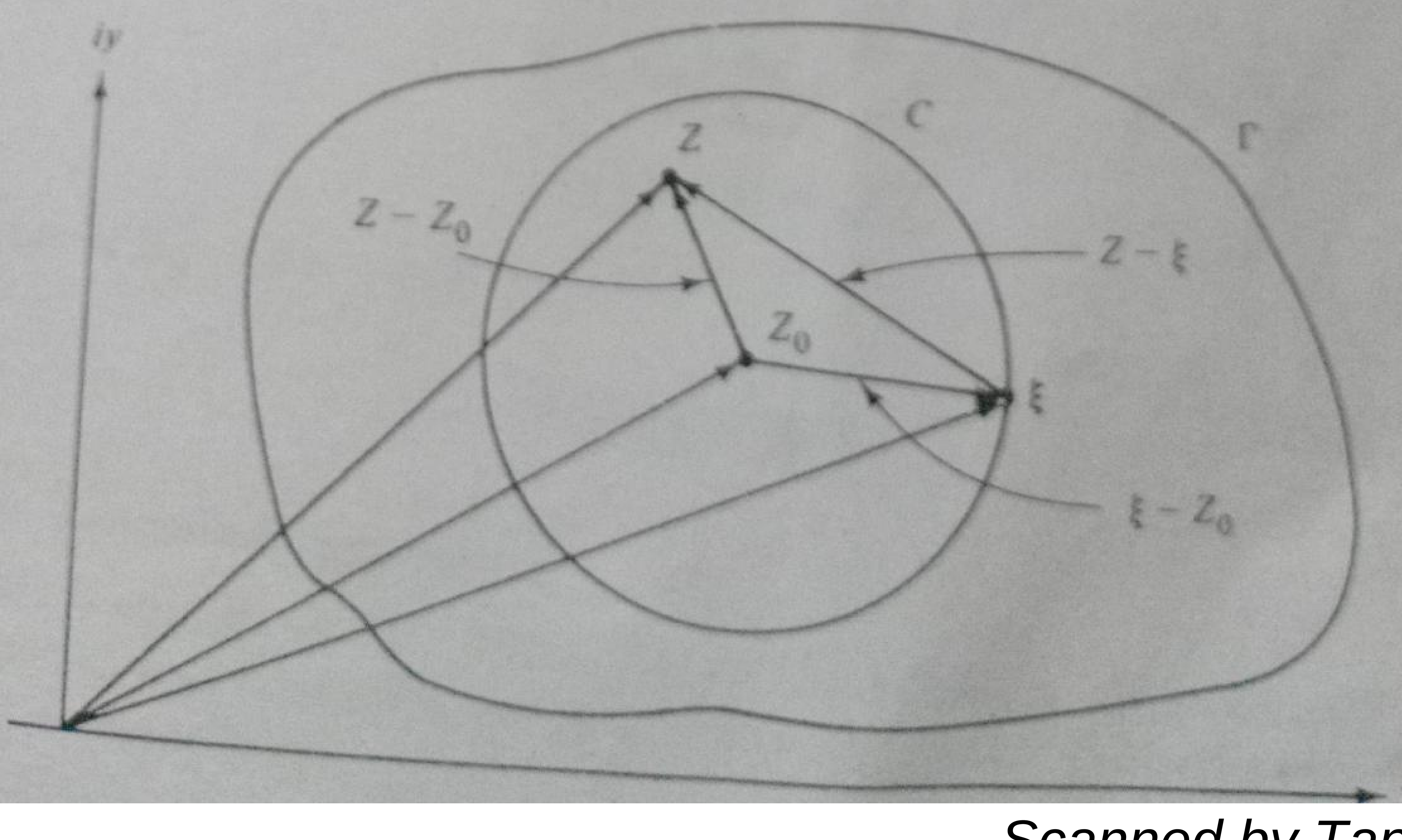
3.4 SERIES EXPANSIONS

The Taylor and Laurent expansions will be developed in this section.

3.4.1 Taylor's Series

If $f(Z)$ is analytic in some region Γ and C is a circle within Γ with center at Z_0, then $f(Z)$ can be expanded in a **Taylor's** (1685–1731) **series.** Here Z is any point interior to C (see Fig. 3.10), that is,

$$
\begin{aligned}
f(Z) &= f(Z_0) + f'(Z_0)(Z - Z_0) + \cdots + \frac{f^{(n)}(Z_0)(Z - Z_0)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(Z - Z_0)^n f^{(n)}(Z_0)}{n!}.
\end{aligned} \tag{3.43}
$$



The series in Eq. (3.43) converges absolutely and uniformly for $|Z - Z_0| < R$, where R is the radius of convergence.

Proof From the Cauchy integral formula, we note that (see Fig. 3.10)

$$f(Z) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)\, d\xi}{\xi - Z} \tag{3.44}$$

where $|Z - Z_0| < R$ and $R = |\xi - Z_0|$. To prove that the expansion holds, a series expansion for $1/(\xi - Z)$ will be developed. For convenience, we set

$$K_T = \frac{Z - Z_0}{\xi - Z_0}. \tag{3.45}$$

On subtracting unity from both sides of Eq. (3.45), we obtain

$$\begin{aligned} 1 - K_T &= 1 - \frac{(Z - Z_0)}{\xi - Z_0} \\ &= \frac{\xi - Z}{\xi - Z_0}. \end{aligned} \tag{3.46}$$

Inverting both sides of Eq. (3.46), we have

$$\begin{aligned} \frac{1}{1 - K_T} &= \frac{\xi - Z_0}{\xi - Z} \\ &= \sum_{n=0}^{\infty} K_T^n \qquad (|K_T| < 1). \end{aligned} \tag{3.47}$$

The required expansion for $1/(\xi - Z)$ is obtained by dividing Eq. (3.47) by $\xi - Z_0$; the result is

$$\begin{aligned} \frac{1}{\xi - Z} &= \frac{1}{\xi - Z_0} \sum_{n=0}^{\infty} \left(\frac{Z - Z_0}{\xi - Z_0}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{(Z - Z_0)^n}{(\xi - Z_0)^{n+1}}. \end{aligned} \tag{3.48}$$

Substituting Eq. (3.48) into Eq. (3.44), we obtain

$$\begin{aligned} \frac{1}{2\pi i} \oint_C \frac{f(\xi)\, d\xi}{\xi - Z} &= \sum_{n=0}^{\infty} \left\{ \frac{1}{2\pi i} \oint_C \frac{f(\xi)(Z - Z_0)^n\, d\xi}{(\xi - Z_0)^{n+1}} \right\} \\ &= \sum_{n=0}^{\infty} \left\{ (Z - Z_0)^n \left(\frac{1}{2\pi i} \oint_C \frac{f(\xi)\, d\xi}{(\xi - Z_0)^{n+1}} \right) \right\} \\ &= \sum_{n=0}^{\infty} \frac{(Z - Z_0)^n f^{(n)}(Z_0)}{n!}. \end{aligned} \tag{3.49}$$

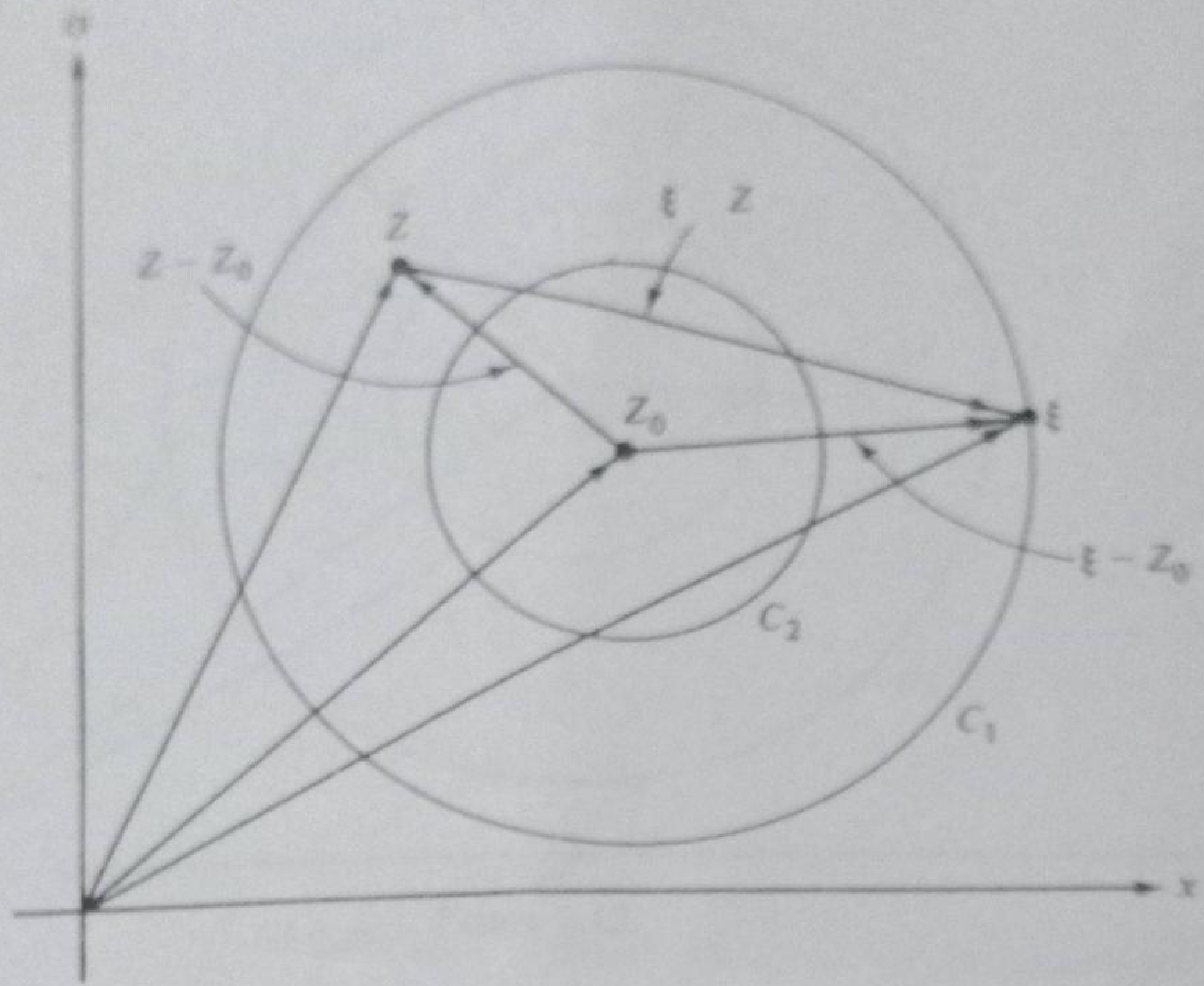


Figure 3.11

where

$$a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(\xi)\,d\xi}{(\xi - Z_0)^{n+1}} \qquad (n = 0, 1, \ldots) \tag{3.51}$$

and

$$b_{n'} = \frac{1}{2\pi i}\oint_{C_1} \frac{f(\xi)\,d\xi}{(\xi - Z_0)^{-n'+1}} \qquad (n' = 1, 2, \ldots). \tag{3.52}$$

The expansion in Eq. (3.50) is called the **Laurent** (1841–1908) **expansion.** In Fig. 3.11, we make a scissor cut from C_1 to C_2 as shown in Fig. 3.12.

Proof On applying the Cauchy integral formula in Fig. 3.12, we find that

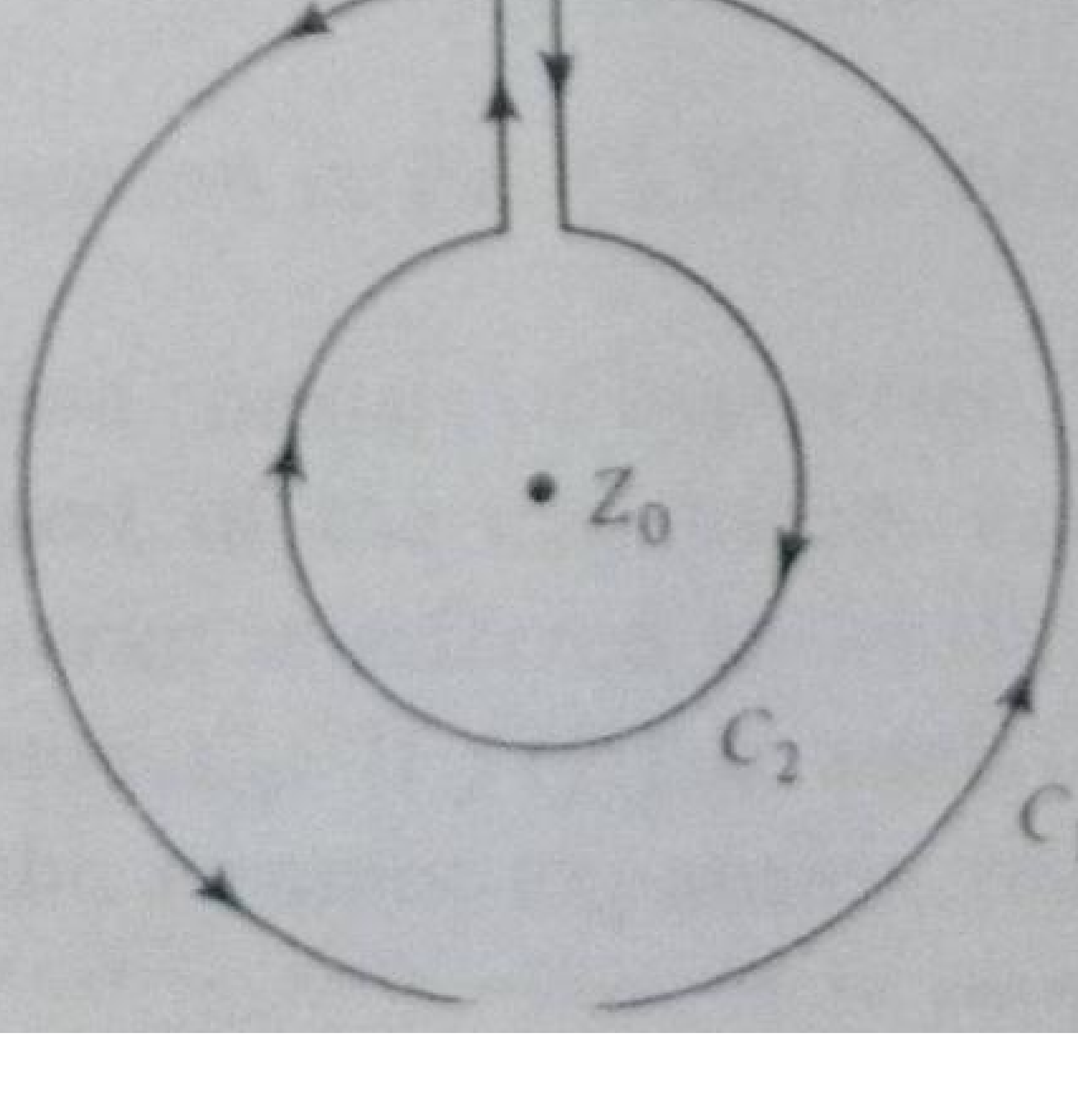


Figure 3.12

$$f(Z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(\xi)\,d\xi}{\xi - Z} - \frac{1}{2\pi i}\oint_{C_2} \frac{f(\xi)\,d\xi}{\xi - Z}$$

$$= f_A(Z) + f_P(Z) \tag{3.53}$$

where

$$f_A(Z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(\xi)\,d\xi}{\xi - Z}$$

and

$$f_P(Z) = \frac{1}{2\pi i}\oint_{C_2} \frac{f(\xi)\,d\xi}{Z - \xi}.$$

Consider $f_A(Z)$ where ξ is on C_1 and Z is inside the region between C_1 and C_2. Using the expansion for $1/(\xi - Z)$ developed in Section 3.4.1, we may write

$$f_A(Z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(\xi)\,d\xi}{\xi - Z} = \sum_{n=0}^{\infty} a_n(Z - Z_0)^n \tag{3.54}$$

where

$$|Z - Z_0| < |\xi - Z_0|$$

$$a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(\xi)\,d\xi}{(\xi - Z_0)^{n+1}}. \tag{3.51'}$$

Note that a_n cannot be represented by $f^{(n)}(Z_0)/n!$ since $f_A(Z)$ is not analytic at $Z = Z_0$ as in the case of the Taylor expansion.

Now consider the integral over C_2, $f_P(Z)$, where it is assumed that ξ is now on C_2 and Z is inside the ring (see Fig. 3.13). We need the appropriate expansion for $1/(Z - \xi)$; it is obtained by the same method used in the Taylor series case. Here we set

$$K_L = \frac{\xi - Z_0}{Z - Z_0}. \tag{3.55}$$

Subtracting unity from both sides of Eq. (3.55) and inverting the resulting equation, we obtain

$$\frac{1}{1 - K_L} = \frac{Z - Z_0}{Z - \xi} = \sum_{n=0}^{\infty} K_L^n$$

or

$$\frac{1}{Z - \xi} = \sum_{n=0}^{\infty} \frac{(\xi - Z_0)^n}{(Z - Z_0)^{n+1}} \tag{3.56}$$

where

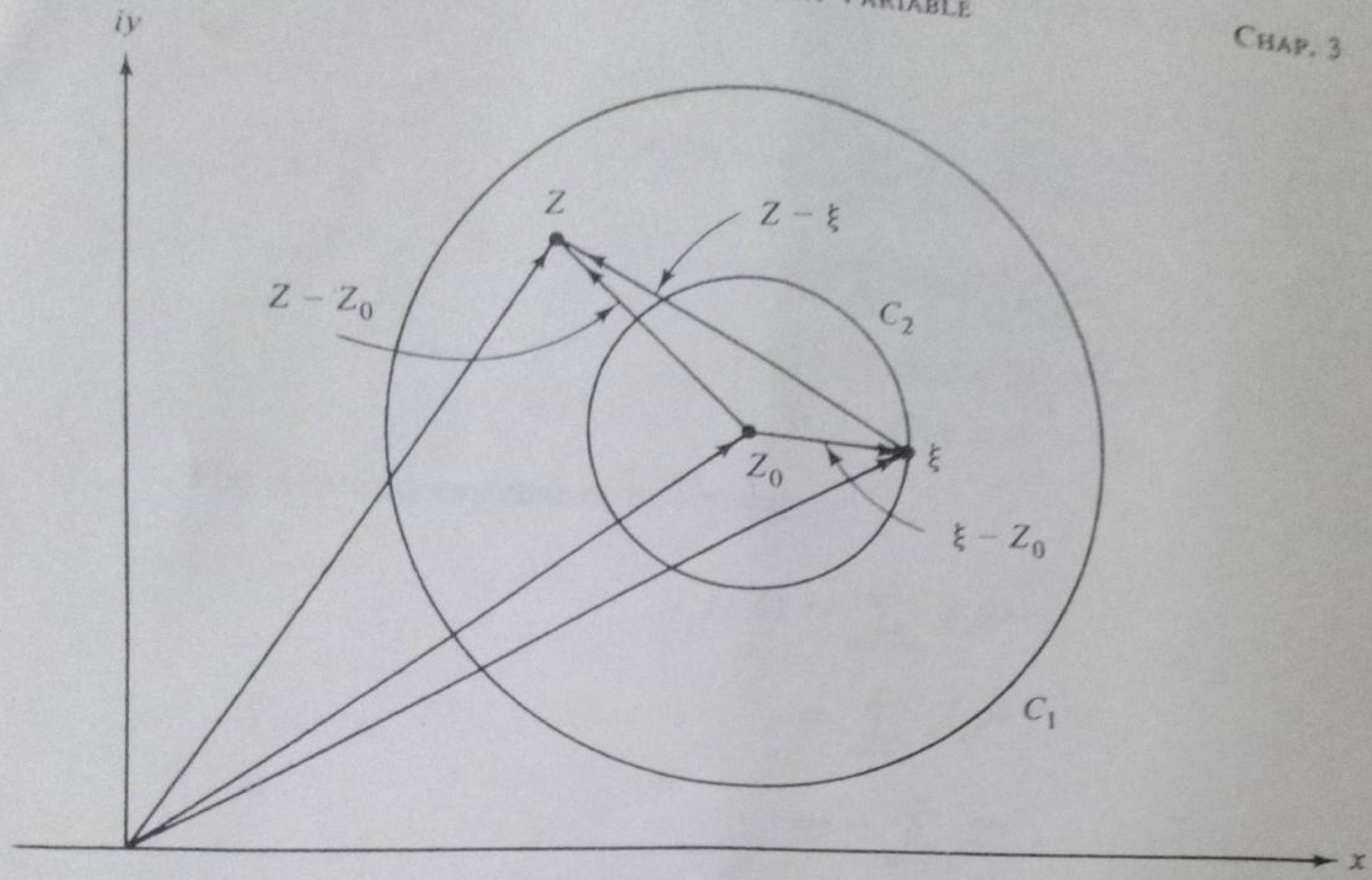


Figure 3.13

On substituting Eq. (3.56) into $f_P(Z)$, we obtain

$$f_P(Z) = \frac{1}{2\pi i}\oint_{C_2} \frac{f(\xi)\,d\xi}{Z-\xi}$$
$$= \sum_{n=0}^{\infty}\left[\frac{1}{(Z-Z_0)^{n+1}}\frac{1}{2\pi i}\oint_{C_2} f(\xi)(\xi-Z_0)^n\,d\xi\right]$$

or

$$f_P(Z) = \sum_{n'=1}^{\infty}\left[\frac{1}{(Z-Z_0)^{n'}}\frac{1}{2\pi i}\oint_{C_2}\frac{f(\xi)\,d\xi}{(\xi-Z_0)^{-n'+1}}\right]$$
$$= \sum_{n'=1}^{\infty}\frac{b_{n'}}{(Z-Z_0)^{n'}} \tag{3.57}$$

where

$$b_{n'} = \frac{1}{2\pi i}\oint_{C_2}\frac{f(\xi)\,d\xi}{(\xi-Z_0)^{-n'+1}}. \tag{3.52'}$$

Hence the Laurent expansion in Eq. (3.50) is established.

The Laurent expansion consists of two series. The first series in Eq. (3.58) is called the **analytic part** of the expansion, and it converges everywhere inside C_1. The second series is referred to as the **principal part**, and it converges everywhere outside C_2. If $f(Z)$ is analytic within C_2, the principal part equals zero, and the Laurent expansion reduces to the Taylor expansion of $f(Z)$ in this case. It is important to note that a real

The expansion in Eq. (3.50), Laurent's expansion, can also be written in the following useful forms:

$$f(Z) = \sum_{n=0}^{\infty} a_n(Z - Z_0)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(Z - Z_0)^n} \tag{3.58}$$

where

$$a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(\xi)\,d\xi}{(\xi - Z_0)^{n+1}} \qquad (n = 0, 1, 2, \ldots)$$

$$a_{-n} = \frac{1}{2\pi i}\oint_{C_2} \frac{f(\xi)\,d\xi}{(\xi - Z_0)^{-n+1}} \qquad (n = 1, 2, 3, \ldots) \tag{3.59}$$

and

$$f(Z) = \sum_{n=-\infty}^{\infty} A_n(Z - Z_0)^n \tag{3.60}$$

where

$$A_n = \frac{1}{2\pi i}\oint_{C} \frac{f(\xi)\,d\xi}{(\xi - Z_0)^{n+1}} \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{3.61}$$

In Eq. (3.61), C is any circle between C_1 and C_2.

EXAMPLE 3.6 By direct evaluation of A_n, find the Laurent expansion for

$$f(Z) = \frac{1}{Z(Z-1)}$$

about $Z_0 = 0$.

Solution Here we have

$$f(Z) = \sum_{n=-\infty}^{\infty} A_n(Z - Z_0)^n$$

$$= \sum_{n=-\infty}^{\infty} A_n Z^n$$

where

$$A_n = \frac{1}{2\pi i}\oint_{C} \frac{f(\xi)\,d\xi}{(\xi - Z_0)^{n+1}}$$

$$= \frac{1}{2\pi i}\oint_{C} \frac{f(\xi)\,d\xi}{\xi^{n+1}}$$

$$= \frac{1}{2\pi i}\oint_{C} \frac{d\xi}{\xi^{n+1}\,\xi(\xi - 1)}$$

$$= -\frac{1}{2\pi i}\oint_{C} \frac{d\xi}{\xi^{n+2}(1 - \xi)}$$

$$= -\frac{1}{2\pi i}\oint \frac{d\xi}{\ldots} \sum^{\infty} \xi^k \qquad (|\xi| < 1)$$

$$A_n = -\frac{1}{2\pi i}\sum_{k=0}^{\infty}\oint_C \frac{d\zeta}{\zeta^{n-k+1}}$$

$$= -\frac{1}{2\pi i}\sum_{k=0}^{\infty} 2\pi i\delta_{n-k+2,1}$$

$$= \begin{cases} -1 & (\text{for } n \geq -1) \\ 0 & (\text{for } n < -1). \end{cases}$$

The required expansion becomes

$$f(Z) = \sum_{n=-\infty}^{\infty} A_n Z^n$$

$$= \sum_{n=-1}^{\infty} A_n Z^n$$

$$= -\sum_{n=-1}^{\infty} Z^n.$$

The problem of developing the Laurent expansion for a function by evaluating the A_n coefficient (or a_n and a_{-n}) is, in general (except for simple functions), tedious. It is sometimes advantageous to use a procedure similar to that illustrated in Examples 3.7 and 3.8 to obtain the required Laurent expansion for a function.

Example 3.7 Find the Laurent expansion for $f(Z) = 1/Z(Z-1)$ by use of a geometric series.

Solution

$$f(Z) = \frac{1}{Z(Z-1)}$$

$$= -\frac{1}{Z}\left(\frac{1}{1-Z}\right)$$

or

$$f(Z) = -\frac{1}{Z}\sum_{n=0}^{\infty} Z^n \qquad (|Z| < 1)$$

$$= -\frac{1}{Z}(1 + Z + Z^2 + \cdots)$$

$$= -\frac{1}{Z} - 1 - Z - Z^2 - \cdots$$

$$= -\sum_{n=-1}^{\infty} Z^n.$$

Note that this result is the same as that obtained in Example 3.6.

EXAMPLE 3.8 Find the Laurent expansion for $f(Z) = 1/Z(Z-2)$ in the regions (a) $0 < |Z| < 2$ and (b) $2 < |Z| < \infty$.

Solution

(a)

$$f(Z) = \frac{1}{Z(Z-2)}$$

$$= -\frac{1}{2}\frac{1}{Z}\left(\frac{1}{1-Z/2}\right)$$

$$= -\frac{1}{2}\frac{1}{Z}\sum_{n=0}^{\infty}\left(\frac{Z}{2}\right)^n \qquad \left(\left|\frac{Z}{2}\right| < 1;\ |Z| < 2\right)$$

or

$$f(Z) = -\frac{1}{2Z}\left(1 + \frac{Z}{2} + \frac{Z^2}{2^2} + \frac{Z^3}{2^3} + \cdots\right)$$

$$= -\frac{1}{2Z} - \frac{1}{2^2} - \frac{Z}{2^3} - \frac{Z^2}{2^4} - \cdots.$$

(b)

$$f(Z) = \frac{1}{Z(Z-2)}$$

$$= \frac{1}{Z^2(1-2/Z)}$$

$$= \frac{1}{Z^2}\sum_{n=0}^{\infty}\left(\frac{2}{Z}\right)^2 \qquad \left(\left|\frac{2}{Z}\right| < 1;\ |Z| > 2\right)$$

$$= \frac{1}{Z^2}\left(1 + \frac{2}{Z} + \frac{2^2}{Z^2} + \cdots\right)$$

$$= \frac{1}{Z^2} + \frac{2}{Z^3} + \frac{2^2}{Z^4} + \cdots.$$

3.5 PROBLEMS

3.1 Show that both ZZ^* and $Z + Z^*$ are real quantities, whereas $Z - Z^*$ is imaginary.

3.2 Find the real and imaginary parts of:

(a) Z^2 (b) $\frac{1}{Z}$

(c) $\frac{Z-1}{Z+1}$ (d) $\frac{1}{Z^2}$.

3.3 Show that $|\cos\theta + i\sin\theta| = 1$.

3.4 Show that $|Z| = |Z^*|$.

3.5 Show that:

(a) $|Z_1 Z_2^*| = |Z_1||Z_2|$

(b) $Z_1 Z_2^* + Z_1^* Z_2 \leq 2|Z_1 Z_2^*|$.