3.3.2 Harmonic Functions

On differentiating Eqs. (3.27) and (3.28) with respect to x and y, respectively, we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\, \partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y\, \partial x}. \tag{3.29}$$

If u and v possess continuous partial derivatives up to second order, Eq. (3.29) leads to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{3.30}$$

Similarly, it can be shown that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{3.31}$$

Therefore u and v are solutions of Laplace's equation, Eqs. (3.30) and (3.31), in two dimensions; they are called **harmonic** (or **conjugate**) **functions.**

EXAMPLE 3.4 (a) Show that $v(x, y) = 3x^2y - y^3$ is harmonic. (b) Find the conjugate function, $u(x, y)$. (c) Find the analytic function $f(Z) = u(x, y) + iv(x, y)$.

Solution

(a) If

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

then v is said to be harmonic.

$$\frac{\partial v}{\partial x} = 6xy \qquad \frac{\partial^2 v}{\partial x^2} = 6y$$

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 \qquad \frac{\partial^2 v}{\partial y^2} = -6y$$

By adding the right-hand sides of $\dfrac{\partial^2 v}{\partial x^2}$ and $\dfrac{\partial^2 v}{\partial y^2}$, we see that v is harmonic.

(b)

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 = \frac{\partial u}{\partial x}$$

Integrating the two above equations with respect to x and y, respectively, we obtain

$$u(x, y) = x^3 - 3y^2x + f(y)$$

and

$$u(x, y) = -3xy^2 + g(x)$$

or

$$x^3 - 3y^2x + f(y) = -3xy^2 + g(x).$$

In the above equation, we must have $f(y) = 0$ and $g(x) = x^3$. The required conjugate function is therefore given by

$$u(x, y) = x^3 - 3y^2x.$$

(c) The corresponding analytic function is given by

$$f(Z) = x^3 - 3y^2x + i(3x^2y - y^3) = Z^3.$$

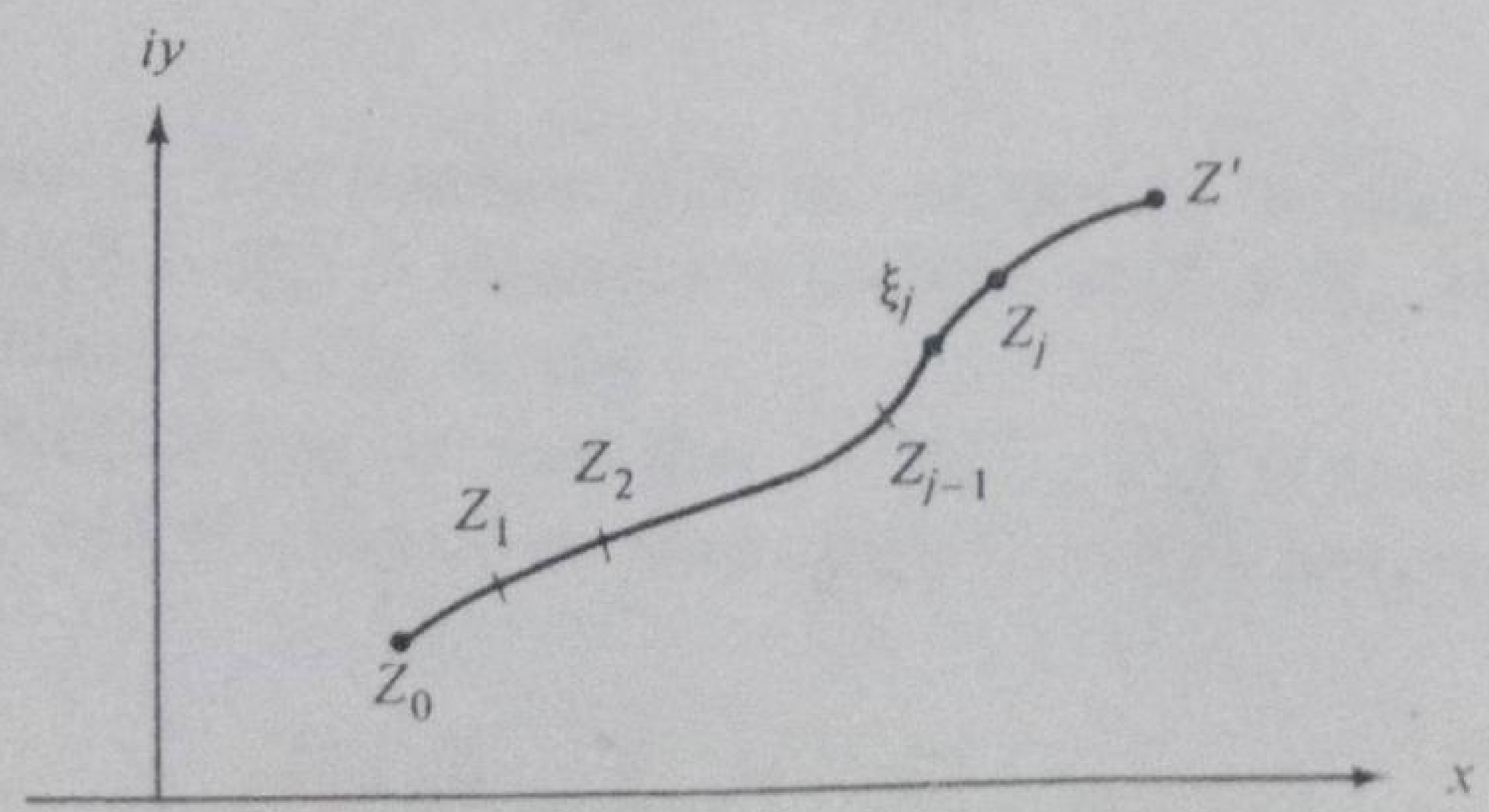
3.3.3 Contour Integrals

The integral (in the Riemannian sense) of the function $f(Z)$ is defined, as in the case of real variables, by

$$\begin{aligned}\int_C f(Z)\,dZ &= \int_{Z_0}^{Z'} f(Z)\,dZ \\ &\equiv \lim_{\substack{n\to\infty \\ |Z_j - Z_{j-1}|\to 0}} \sum_{j=1}^{n} f(\xi_j)(Z_j - Z_{j-1}).\end{aligned} \tag{3.32}$$

where the path C has been divided into n segments and ξ_j is some point on the curve between Z_{j-1} and Z_j (see Fig. 3.7).

In complex variable theory, $\int_C f(Z)\,dZ$ is called a **contour integral** of $f(Z)$ along the contour C from Z_0 to Z'.



3.3.4 Cauchy's Integral Theorem

THEOREM If $f(Z)$ is analytic and its partial derivatives are continuous throughout some simply connected region, then for every closed path C within this region

$$\oint_C f(Z)\,dZ = 0. \tag{3.33}$$

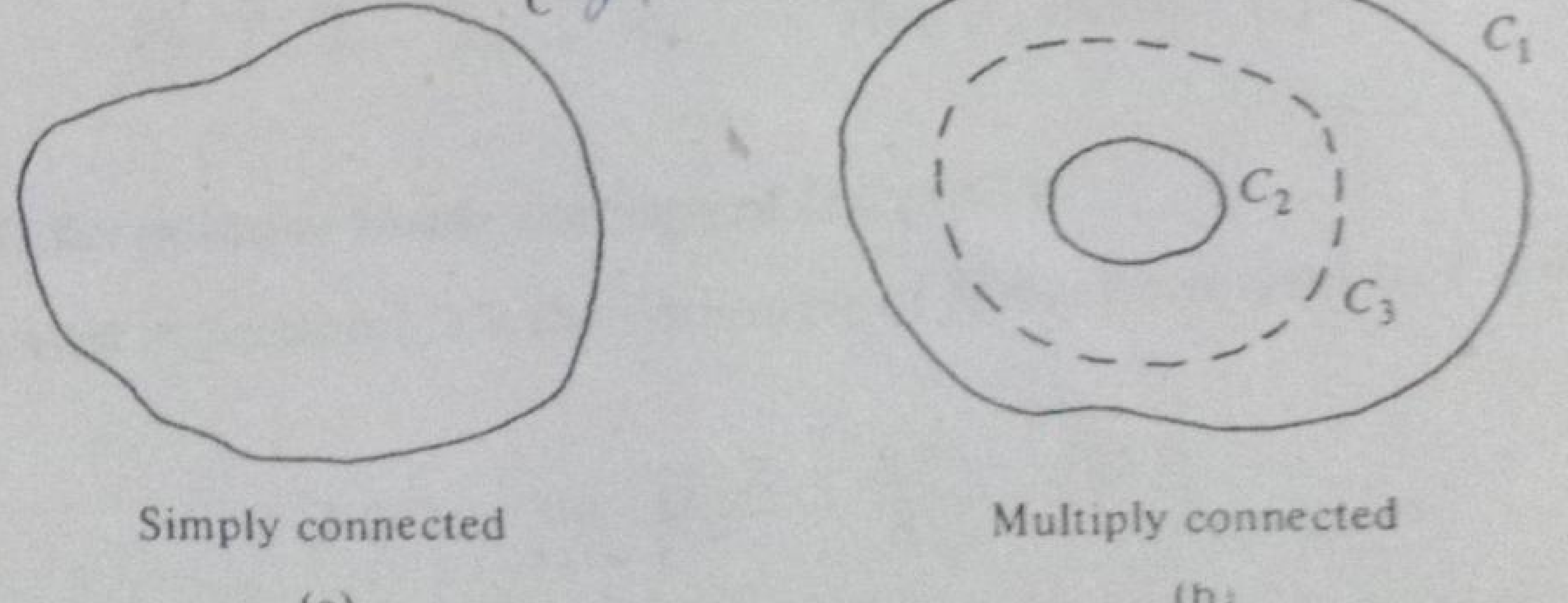
The symbol $\oint$ means the integral is around a closed path, and the symbol $\oint$ indicates that the path is traversed in a positive (counterclockwise) manner. Our convention will be to traverse the path in such a way that the region of interest lies to the left. Equation (3.33) is called **Cauchy's integral theorem.**

The following are three equivalent definitions of a **simply connected** region: A region in the complex plane is called simply connected if (1) all closed paths within the region contain only points that belong to the region, (2) all closed paths within the region can be shrunk to a point, and (3) it has the property that every scissor cut starting at an arbitrary point, Z_1, on the boundary and finishing at another point, Z_2, on the boundary separates the region into unconnected (two) pieces. All other regions are said to be **multiply connected.** This classification is illustrated in Fig. 3.8. A closed curve drawn within C, Fig. 3.8(a), contains only points that belong to C. However, all points within C_3, Fig. 3.8(b), do not belong to the region between C_1 and C_2. Therefore the region within C is simply connected, and that between C_1 and C_2 is multiply connected.

The curl theorem will be used in developing a proof of Cauchy's integral theorem. On expanding the left-hand side of Eq. (3.33), we get

$$\begin{aligned}\oint_C f(Z)\,dZ &= \oint_C (u + iv)(dx + i\,dy)\\ &= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy).\end{aligned} \tag{3.34}$$

∬ Fx C σ ∬ σ



Simply connected (a) Multiply connected (b)

The curl theorem due to Stokes (Section 1.4.4)

$$\iint_{\sigma} \nabla \times \mathbf{F} \cdot d\boldsymbol{\sigma} = \oint_{\lambda} \mathbf{F} \cdot d\boldsymbol{\lambda}$$

in two dimensions becomes

$$\iint_{\sigma} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dx\, dy = \oint_{\lambda} (F_x\, dx + F_y\, dy) \tag{3.35}$$

where F_x and F_y are any two functions with continuous partial derivatives. Substituting Eq. (3.35) into the first and second terms on the right-hand side of Eq. (3.34) separately and applying the Cauchy-Riemann conditions, we obtain

$$\oint_C f(Z)\, dZ = -\iint_{\sigma} \underbrace{\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)}_{=0} dx\, dy + i \iint_{\sigma} \underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{=0} dx\, dy = 0. \tag{3.36}$$

It is clear that Cauchy's integral theorem tells us the integral of $f(Z)$ around a closed path in a simply connected region is independent of the path.

A less restrictive theorem due to Goursat (1858–1936) which eliminates the continuous partial derivative requirement can also be proved. The inverse of Cauchy's theorem is known as **Morera's** (1856–1909) **theorem.**

3.3.5 Cauchy's Integral Formula

Another important and extremely useful relation concerning the integral of a function of a complex variable is **Cauchy's integral formula.** It can be written as

$$\oint_C \frac{f(Z)\, dZ}{Z - Z_0} = 2\pi i f(Z_0) \tag{3.37}$$

where Z_0 is within C. The function $f(Z)$ is assumed to be analytic within C; however, $f(Z)/(Z - Z_0)$ is clearly not analytic at $Z = Z_0$. For $r \to 0$, Fig. 3.9(a) is equivalent to Fig. 3.9(b).

From Fig. 3.9(b), it can be seen that $f(Z)/(Z - Z_0)$ is analytic in the region between C and C'. Hence we may apply Cauchy's integral theorem and obtain

$$\oint_{C} \frac{f(Z)\, dZ}{Z - Z_0} + \oint_{C'} \frac{f(Z)\, dZ}{Z - Z_0} = 0$$

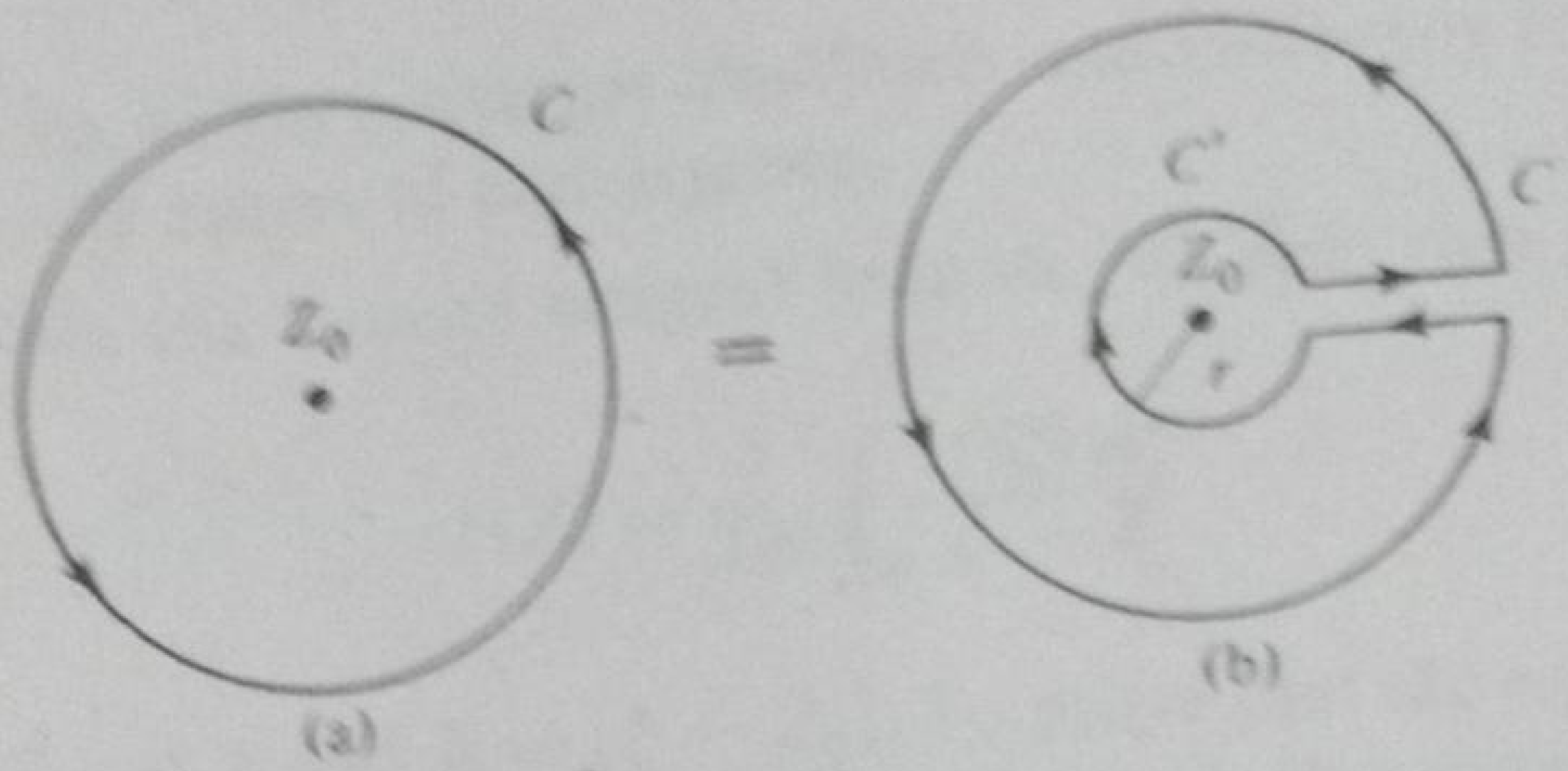


Figure 3.9

or

$$\oint_C \frac{f(Z)\,dZ}{Z-Z_0} = \oint_{C'} \frac{f(Z)\,dZ}{Z-Z_0}. \tag{3.38}$$

Around the path C', we set

$$Z - Z_0 = re^{i\theta}; \qquad dZ = ire^{i\theta}\,d\theta. \tag{3.39}$$

Substituting Eq. (3.39) into Eq. (3.38), we obtain

$$\begin{aligned}\oint_C \frac{f(Z)\,dZ}{Z-Z_0} &= \int_0^{2\pi} \frac{f(Z_0 + re^{i\theta})ire^{i\theta}\,d\theta}{re^{i\theta}} \\ &= i\int_0^{2\pi} f(Z_0 + re^{i\theta})\,d\theta \\ &= 2\pi i f(Z_0)\end{aligned}$$

as $r \to 0$. $Z \to Z_0$

for $r \to 0$. Hence the formula in Eq. (3.37) is established. In summary, we may write

$$\frac{1}{2\pi i}\oint_C \frac{f(Z)\,dZ}{Z-Z_0} = \begin{cases} f(Z_0) & (\text{for } Z_0 \text{ inside } C) \\ 0 & (\text{for } Z_0 \text{ outside } C). \end{cases} \tag{3.40}$$

3.3.6 Differentiation Inside the Sign of Integration

As indicated in Section 3.3.1, the derivative of $f(Z)$ with respect to Z is defined by

$$f'(Z) = \lim_{\Delta Z \to 0} \left\{ \frac{f(Z+\Delta Z) - f(Z)}{\Delta Z} \right\}. \tag{3.41}$$

Applying Cauchy's integral formula to Eq. (3.41), we obtain