Lec 3: Absolute convergence

Example The alternating harmonic series given by

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots$$

is convergent.

Definition A series $\sum x_n$ is absolutely convergent if $\sum |x_n|$ is convergent.

A series is conditionally convergent if it is convergent but not absolutely convergent.

* **Note** It is trivial that a series of non-negative terms is absolute convergent if and only if it is convergent.

Thm 3.1 If a series is absolutely convergent then it is convergent.

Proof Since $\sum |x_n|$ is convergent, therefore for any $\varepsilon > 0$, $\exists\ M(\varepsilon) \in \mathbb{N}$ such that

$$m > n \geq M(\varepsilon) \Rightarrow |x_{n+1}| + |x_{n+2}| + \cdots + |x_m| < \varepsilon$$

Hence $\quad |x_{n+1} + x_{n+2} + \cdots + x_m| \leq |x_{n+1}| + |x_{n+2}| + \cdots + |x_m| < \varepsilon$

Therefore $\sum x_n$ is convergent.

Grouping of series From a given series $\sum x_n$ we can construct a new series $\sum y_n$ by grouping (by inserting parentheses) some finite number of terms.

Example : If $\sum x_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ we may have the new series

$$\sum y_n = 1 - \frac{1}{2} + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6} + \frac{1}{7}\right) - \frac{1}{8} + \cdots$$

Thm 3.2 If a series $\sum x_n$ is convergent then any series obtained ~~by~~ from it by grouping the terms is also convergent and to the same value.

Proof Let $\sum y_k$ be the series obtained from $\sum x_n$ by grouping ~~some~~ its terms. Suppose we have

$$y_1 = x_1 + \cdots + x_{k_1}, \quad y_2 = x_{k_1+1} + \cdots + x_{k_2}, \cdots$$

Let s_n denote the nth partial sum of $\sum x_n$ and t_k denote the kth partial sum of $\sum y_k$. Then we have

$$t_1 = y_1 = s_{k_1}, \quad t_2 = y_1 + y_2 = s_{k_2}, \cdots$$

Thus the sequence $\langle t_k \rangle$ is actually a ~~partial~~ sub-sequence of the sequence $\langle s_n \rangle$.

Since $\langle s_n \rangle$ is convergent, therefore $\langle t_k \rangle$ must be convergent and hence $\sum y_n$ is convergent.

* **Note** Converse of the above theorem is false.

For example consider the sequence $\sum_{n=1}^{\infty} (-1)^n$

Definition. A series $\sum y_k$ is a rearrangement of the series $\sum x_n$ if there is a bijection $f: \mathbb{N} \longrightarrow \mathbb{N}$ such that $y_k = x_{f(k)}$ $\forall$ $k \in \mathbb{N}$.

Theorem 3.3 (Re-arrangement theorem) Let $\sum x_n$ be an absolutely convergent series. Then any re-arrangement $\sum y_k$ of $\sum x_n$ converges to the same value.

Proof Suppose $\sum x_n = x$ i.e. $\sum x_n$ converges to x

Hence for any $\varepsilon > 0$ $\exists$ $N_0 \in \mathbb{N}$ such that $n, m \geq N_0 \Rightarrow$

$$|x_{N_0+1}| + |x_{N_0+2}| + \cdots + |x_m| < \varepsilon$$

i.e. $$\sum_{k=N_0+1}^{m} |x_k| < \varepsilon$$

Also if $\langle s_n \rangle$ is a sequence of partial sums of $\sum x_n$ then

$$\lim_{n \to \infty} s_n = x$$

Hence for the same ε we can have $N_1 \in \mathbb{N}$ such that

$$|x - s_n| < \varepsilon \qquad \forall \ n \geq N_1$$

Let $N = \max\{N_0, N_1\}$

Then $\forall$ $n, m > N$ we have

$$|x - s_n| < \varepsilon \quad \text{and} \quad \sum_{k=N+1}^{m} |x_k| < \varepsilon$$

Let $\langle t_k \rangle$ be sequence of partial sums of the re-arranged series $\sum y_k$

Let $M \in \mathbb{N}$ be such that all of terms $x_1, x_2, \dots, x_N$ are contained as summands in $t_M = y_1 + y_2 + \cdots + y_M$

For $q \geq M$ we have $t_q - s_q$ is the sum of a finite number of terms x_k such that $k > N$. Hence for some $m > N$ we have

$$|t_q - s_m| \leq \sum_{k=N+1}^{q} |x_k| < \varepsilon$$

Hence for $q \geq M$ we have

$$|t_q - x| \leq |t_q - s_m| + |s_m - x| < \varepsilon + \varepsilon = 2\varepsilon$$

Since, ε is arbitrary, hence $\langle t_k \rangle$ converges to x which implies that $\sum y_k$ converges to x.

Tests for absolute convergence

Theorem 3.4 (Limit comparison test II) Let $\sum x_n$ and $\sum y_n$ be two infinite series and suppose the following limit exists in $\mathbb{R}$:

$$r = \lim_{n\to\infty} \left|\frac{x_n}{y_n}\right|$$

(a) If $r \neq 0$ then $\sum x_n$ is absolutely convergent if and only if $\sum y_n$ is absolutely convergent.

(b) If $r = 0$ and if $\sum y_n$ is absolutely convergent then $\sum x_n$ is absolutely convergent.

Thm 3.5 (Cauchy's root test) Let $\sum x_n$ be an infinite series.

(a) If $\exists\, r \in \mathbb{R}$ with $r<1$ and $K \in \mathbb{N}$ such that

$$|x_n|^{1/n} \leq r \qquad \text{for } n \geq K$$

then the series $\sum x_n$ is absolutely convergent.

(b) If there exists $K \in \mathbb{N}$ such that

$$|x_n|^{1/n} \geq 1 \qquad \text{for } n \geq K$$

then the series $\sum x_n$ is divergent

Proof (a) Suppose for $n \geq K$ and $r<1$

$$|x_n|^{1/n} \leq r$$

$$\Rightarrow |x_n| \leq r^{1/n}$$

(b) Suppose for $n \geq K$

$$|x_n|^{1/n} \geq 1$$

$$\Rightarrow |x_n| \geq 1$$

Hence $\lim\limits_{n \to \infty} x_n \neq 0$

Therefore $\sum x_n$ is divergent

<u>Corollary 3.6</u> Let $\sum x_n$ be an infinite series and suppose the limit

$$r = \lim_{n \to \infty} |x_n|^{1/n}$$

exists in $\mathbb{R}$. Then

(a) $\sum x_n$ is absolutely convergent if $r<1$

(b) $\sum x_n$ is divergent if $r>1$

(c) No conclusion is possible if $r=1$.

<u>Proof</u> (a) Since $\lim\limits_{n \to \infty} |x_n|^{1/n} = r < 1$

therefore there exists $r_1 \in \mathbb{R}$ with $r < r_1 < 1$ and $K \in \mathbb{N}$ such that

$$|x_n|^{1/n} \leq r_1 \qquad \forall\ n \geq K$$

Hence $\sum x_n$ is absolutely convergent.

(b) If $r>1$ then there exists $K \in \mathbb{N}$ such that

$$|x_n|^{1/n} > 1 \qquad \forall\ n \geq K$$

Hence $\sum x_n$ is divergent.