Exact Differential Eqn.

Defn. An eqn. which always can be obtained from its primitive (i.e. Gen. Soln.) directly by differentiation, without any subsequent multiplication, elimination etc.

Necessary & Sufficient condition for an eqn. to be exact:

To find the necessary and sufficient condition for a differential eqn. of first degree being exact.

Soln. Let the eqn. be

$$M + N\frac{dy}{dx} = 0 \longrightarrow (1)$$

where M and N are functions of x and y.

Let

$$u = c \longrightarrow (2)$$

be its primitive (i.e. General soln.)

Necessary part: If (1) is exact, it can be obtained by directly differentiating its primitive.

Differentiating (2), we have

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = 0 \longrightarrow (3)$$

Comparing (1) & (3), we get

$$M = \frac{\partial u}{\partial x} \text{ and } N = \frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}, \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

Hence the necessary condition is

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}.$$

Sufficient part:

Let

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Then we have to show that

$$M + N\frac{\partial y}{\partial x} = 0 \quad \text{or,} \quad Mdx + Ndy = 0 \text{ is exact.}$$

Let $\int M dx = U$.

$$\therefore \frac{\partial U}{\partial x} = M \text{ so that}$$

$$\frac{\partial^2 U}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \left[\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right]$$

i.e. $$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right)$$

Integrating, we get

$$N = \frac{\partial U}{\partial y} + f(y)$$

where $f(y)$ is a function of y free from x.

$$\therefore M + N\frac{dy}{dx} = \frac{\partial U}{\partial x} + \left[\frac{\partial U}{\partial y} + f(y)\right]\frac{dy}{dx}$$

$$= \left(\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y}\frac{dy}{dx}\right) + f(y)\frac{dy}{dx}$$

or,

$$M+N\frac{dy}{dx} = \frac{d}{dx}\left[U+\int f(y)\frac{dy}{dx}\,dx\right],$$

$$\left[\because \frac{dU}{dx} = \frac{\partial U}{\partial x}\frac{dx}{dx} + \frac{\partial U}{\partial y}\cdot\frac{dy}{dx} = \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y}\cdot\frac{dy}{dx}\right]$$

$$= \frac{d}{dx}\left[U+F(y)\right]$$

This shows that $M+N\frac{dy}{dx} = 0$ is exact eqn.

Working rule:

If the eqn $M\,dx + N\,dy = 0$ satisfies the condn

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

then it is exact. Then the Gen. soln is

$$\boxed{\int_{y=\text{const}} M\,dx + \int (\text{terms free from } x \text{ in } N)\,dy = C}$$

Ex. Solve $(x^2-2xy+3y^2)dx + (4y^3+6xy-x^2)dy = 0$.

Soln. Here $M = x^2-2xy+3y^2$, $N = 4y^3+6xy-x^2$.

Now, $\frac{\partial M}{\partial y} = -2x+6y$, $\frac{\partial N}{\partial x} = 6y-2x$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Hence given eqn. is exact.

$\therefore$ G.S. is

$$\int_{y\text{-const}} M\,dx + \int (\text{terms free from } x \text{ in } N)\,dy = c$$

$$\Rightarrow \int_{y\text{-const.}} (x^2-2xy+3y^2)dx + \int 4y^3\,dy = c$$

$$\Rightarrow \frac{x^3}{3} - x^2y + 3xy^2 + y^4 = c \longleftarrow \text{Ans.}$$