Interpolation with equal Intervals

§.1. Newton-Gregory's forward interpolation formula:

Let $f(a), f(a+h), f(a+2h), \ldots, f(a+nh)$ be the $(n+1)$ values of the function $y=f(x)$ corresponding to the equally spaced values $a, a+h, a+2h, \ldots, a+nh$ of x.

Now, we shall find an approximate representation of $f(x)$ in the form of a polynomial.

Let $P_n(x)$ be a poly. of degree n given by

$$P_n(x) = A_0 + A_1(x-a) + A_2(x-a)(x-a-h) + A_3(x-a)(x-a-h)(x-a-2h) + \cdots + A_n(x-a)(x-a-h)(x-a-2h)\cdots(x-a-\overline{n-1}h) \quad \longrightarrow (1)$$

where $A_0, A_1, A_2, \ldots, A_n$ are constant to be determined s.t.

$$\left.\begin{aligned} P_n(a) &= f(a), \\ P_n(a+h) &= f(a+h) \\ &\cdots\cdots \\ P_n(a+nh) &= f(a+nh). \end{aligned}\right\}$$

Now, putting $x = a, a+h, a+2h, \ldots$ successively in (1), we get

$$P_n(a) = A_0$$

$$\Rightarrow f(a) = A_0$$

Also,

$$P_n(a+h) = A_0 + A_1 h$$

$$\Rightarrow f(a+h) = f(a) + A_1 h$$

$$\Rightarrow A_1 h = f(a+h) - f(a) = \Delta f(a)$$

$$\therefore A_1 = \frac{\Delta f(a)}{h}$$

Again,

$$P_n(a+2h) = A_0 + A_1 2h + A_2 2h.h$$

$$\Rightarrow f(a+2h) = f(a) + \frac{f(a+h) - f(a)}{h} \cdot 2h + A_2 h^2 \lfloor 2$$

$$\Rightarrow A_2 h^2 \lfloor 2 = f(a+2h) - 2f(a+h) + f(a)$$

$$= \Delta^2 f(a)$$

$$\therefore A_2 = \frac{\Delta^2 f(a)}{h^2 \lfloor 2}$$

Similarly, $A_3 = \frac{\Delta^3 f(a)}{h^3 \lfloor 3}, \ldots, A_n = \frac{\Delta^n f(a)}{\lfloor n \, h^n}$.

$\therefore (1) \Rightarrow$

$$f(x) \equiv P_n(x) = f(a) + \frac{\Delta f(a)}{h}(x-a)$$

$$+ \frac{\Delta^2 f(a)}{\lfloor 2 \, h^2}(x-a)(x-a-h)$$

$$+ \frac{\Delta^3 f(a)}{\lfloor 3 \, h^3}(x-a)(x-a-h)(x-a-2h)$$

$$+ \ldots\ldots\ldots$$

$$+ \frac{\Delta^n f(a)}{\lfloor n \, h^n}(x-a)(x-a-h)(x-a-2h)\ldots\ldots(x-a-\overline{n-1}h) \longrightarrow (2)$$

This is called Newton's (or Newton–Gregorey's) forward interpolation formula in terms of x.

Scanned with CamScanner

Now, let us put

$$\boxed{u = \frac{x-a}{h}} \quad \text{i.e. } x = a + hu$$

$\therefore$ (2) $\Rightarrow$

$$f(x) = f(a) + \frac{\Delta f(a)}{h} hu + \frac{\Delta^2 f(a)}{\lfloor 2 \, h^2} \cdot hu(hu-h)$$

$$+ \cdots\cdots + \frac{\Delta^n f(a)}{\lfloor n \, h^n} hu(hu-h)\cdots(hu - \overline{n-1}\,h)$$

$$\Rightarrow f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{\lfloor 2} \Delta^2 f(a)$$

$$+ \frac{u(u-1)(u-2)}{\lfloor 3} \Delta^3 f(a) + \cdots$$

$$\cdots + \frac{u(u-1)(u-2)(u-3)\cdots(u-n+1)}{\lfloor n} \Delta^n f(a)$$

$\longrightarrow (3)$

This is the form in which Newton-Gregory's formula for forward interpolation is usually written. This formula is used mainly for interpolating (i.e. for determining) the values of $y = f(x)$ near the begining of a set of tabular values. #

$\boxed{\text{Note}}$: Writing $f(a) = y_0$, $f(a+h) = y_1$, $f(a+2h) = y_2, \ldots,$ $f(a+nh) = y_n$,

(3) $\Rightarrow$

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{\lfloor 2} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{\lfloor 3} \Delta^3 y_0$$

$$+ \cdots + \frac{u(u-1)(u-2)\cdots(u-n+1)}{\lfloor n} \Delta^n y_0 \longrightarrow (4)$$

which is also Newton's forward interpolation formula #

Scanned with CamScanner

Cor: Putting $h=1$, $a=0$, we get $x=u$.

$\therefore$ (3) $\Rightarrow$

$$f(x) = f(0) + x\,\Delta f(0) + \frac{x(x-1)}{\lfloor 2}\Delta^2 f(0) + \cdots + \frac{x(x-1)(x-2)\cdots(x-n+1)}{\lfloor n}\Delta^n f(0)$$

Here the values of x are $0, 1, 2, 3, \ldots, n$.

Ex. Find the form of the function $f(x)$ for the following:

x:	0	1	2	3	4
$f(x)$:	1	3	5	7	9

Sol^n: Let us first form the following diff. table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	1				
		2			
			0		
1	3			0	
		2			0
			0		
2	5			0	
		2			
			0		
3	7				
		2			
4	9				

Here $a=0$, $h=1$. $\therefore x = u$ $\left(\because u = \frac{x-a}{h}\right)$

$\therefore$ By Newton's forward interpolation formula, we get

$$f(x) = f(0) + x\,\Delta f(0) + \frac{x(x-1)}{\lfloor 2}\Delta^2 f(0) + \cdots$$

$$\Rightarrow f(x) = 1 + x\cdot 2 + 0 = 2x+1 \leftarrow \text{Ans}$$

Note: By prev. formula, $f(x) = f(0) + {}^{x}C_1\,\Delta f(0) + {}^{x}C_2\,\Delta^2 f(0) + \cdots + {}^{x}C_n\,\Delta^n f(0)$

$= f(0) + x^{(1)}\cdot 2 + x^{(2)}\cdot 0 + 0 + \cdots + 0$

$= 1 + x\cdot 2 + 0 + \cdots + 0 = 1 + 2x.$